Towards a renormalization theory for quasi-periodically forced one dimensional maps I. Existence of reducibility loss bifurcations*

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Abstract

We propose an extension of the one dimensional (doubling) renormalization operator to the case of maps on the cylinder. The kind of maps considered are commonly referred as quasi-periodic forced one dimensional maps. We prove that the fixed point of the one dimensional renormalization operator extends to a fixed point of the quasi-periodic forced renormalization operator. We also prove that the operator is differentiable around the fixed point and we study its derivative. Then we consider a two parametric family of quasi-periodically forced maps which is a unimodal one dimensional map with a full cascade of period doubling bifurcations plus a quasi-periodic perturbation. For one dimensional maps it is well known that between one period doubling and the next one there exists a parameter value where the 2^n-periodic orbit is superatracting. Under appropriate hypotheses, we prove that the two parameter family has two curves of reducibility loss bifurcation around these points.

Contents

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1 Introduction

This is the first of a series of papers (together with [22, 23]) proposing an extension of the one dimensional renormalization theory for the case of quasi-periodic forced maps. Each of these papers is self contained, but highly interrelated with the others. A more detailed exposition can be found in [20]. In this paper we give a concrete definition of the operator to the case of quasi-periodic maps and we use it to prove the existence of reducibility loss bifurcations when the coupling parameter goes to zero, like the ones observed in [12, 21] for the Forced Logistic map. In [22] we will use the results developed here to study the asymptotic behavior of these bifurcations when the period of the attracting set goes to infinity. Our quasi-periodic extension of the renormalization operator is not complete in the sense that several conjectures must be assumed. In [23] we include the numerical evidence which support these conjectures and we show that the theoretical results agree with the behavior observed numerically.

The classic one dimensional renormalization theory was motivated to explain the cascades of period doubling bifurcations. The paradigmatic example in the case of unimodal maps is the Logistic Map, but the properties of renormalization and universality are also observable in a wider class of maps. Concretely, given a typical one parametric family for unimodal maps $\{f_\alpha\}_{\alpha\in I}$ one observes numerically that there exists a sequence of parameter values $\{d_n\}_{n\in\mathbb{N}} \subset I$ such that the attracting periodic orbit of the map undergoes a period doubling bifurcation. Between one period doubling and the next one there exists also a parameter value s_n, for which the critical point of f_{s_n} is a periodic orbit with period 2^n. One can also observe that

$$\lim_{n\to\infty} \frac{d_n - d_{n-1}}{d_{n+1} - d_n} = \lim_{n\to\infty} \frac{s_n - s_{n-1}}{s_{n+1} - s_n} = \boldsymbol{\delta} = \texttt{4.66920...} \tag{1}$$

Moreover, the constant $\boldsymbol{\delta}$ is universal, in the sense that for any family of unimodal maps with a quadratic turning point having a cascade of period doubling bifurcations, one obtains the same ratio $\boldsymbol{\delta}$. For technical reason the discussion is typically focussed around the values s_n.

The renormalization theory for unimodal one dimensional maps was originated by the seminal works of Feigenbaum ([7, 8]) and Collet and Tresser ([25]) who independently proposed the renormalization operator to explain the universal behavior observed in the cascades of bifurcations of one dimensional maps, see [4] for a review. Let us do a quick summary of the theory. The (doubling) renormalization operator, which is denoted by $\mathcal{R}$, is defined in the space of unimodal maps as the self composition of the map composed with a change of scale (see subsection 2.1 for more details). There are some basic assumptions on the dynamics of the operator $\mathcal{R}$ (known as the Feigenbaum conjectures) which give a suitable explanation to the universality described before. The first of these conjectures is that the operator has a fixed point Φ and it is differentiable in a neighborhood of Φ. The second conjecture is that the spectrum of $D\mathcal{R}(\Phi)$ has a a unique real eigenvalue $\boldsymbol{\delta}$ bigger than one, and the rest of eigenvalues are strictly smaller

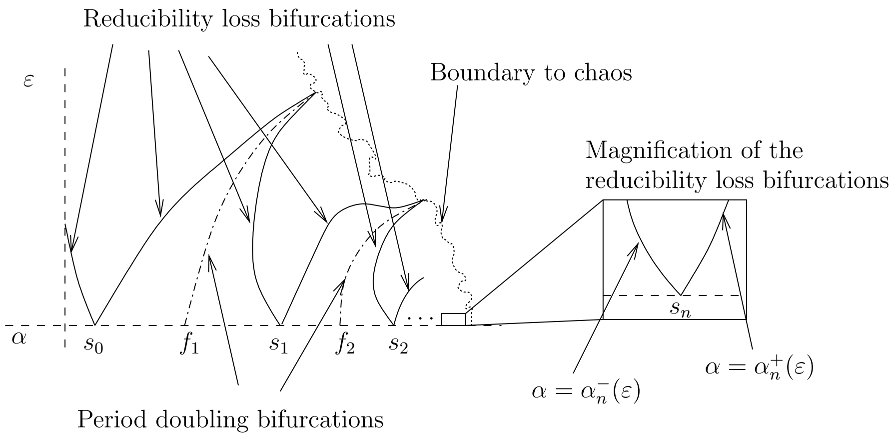

Figure 1: Schematic representation of the bifurcations diagram of the Forced Logistic Map, see [12] for the numeric computation of this diagram.

than one. Then one has that the unstable manifold $W^u(\Phi, \mathcal{R})$ has dimension one and the stable manifold $W^s(\Phi, \mathcal{R})$ has codimension one.

On the other hand, one has that the values $\alpha = s_n$ where the critical point of a map f_{s_n} has period 2^n correspond to the parameter values where the family $\{f_\alpha\}_{\alpha \in I \subset \mathbb{R}}$ intersects certain codimension one manifolds Σ_n. Moreover one has that $\mathcal{R}(\Sigma_n) \subset \Sigma_{n-1}$. The third conjecture claims that these manifolds intersect transversally $W^u(\Phi, \mathcal{R})$. Then one has that they accumulate to $W^s(\Phi, \mathcal{R})$ with ratio $\boldsymbol{\delta}$ in a neighborhood U of Φ. Then, for any one dimensional family $\{f_\alpha\}_{\alpha \in I}$ intersecting $W^s(\Phi, \mathcal{R}) \cap U$ transversally, the family has a sequence $\{s_n\}_{n \in \mathbb{N}}$ of parameters where the critical point of the map f_{s_n} is periodic with period 2^n. Hence, applying the λ-lemma, this parameter values satisfy the asymptotic behavior given by the equation (1), with $\boldsymbol{\delta}$ the unstable eigenvalue of $D\mathcal{R}(\Phi)$.

The first proofs of the Feigenbaum conjectures were done with computer assistance ([17, 6]). Later on completely conceptual proofs appeared ([24, 19]), all of them for the case of analytic maps. For studies of the operator in the C^r context see [3] and [2]. Our extension to the quasi-periodic case does not cover all the theory exposed in the cited works.

The paradigmatic example in this paper is the Forced Logistic Map (FLM for short). Nevertheless the obtained results are applicable to a wider class of maps. The FLM is a map in the cylinder $\mathbb{T} \times \mathbb{R}$ defined as

$$\left.\begin{array}{rclcl} \bar{\theta} & = & r_\omega(\theta) & = & \theta + \omega, \\ \bar{x} & = & f_{\alpha,\varepsilon}(\theta, x) & = & \alpha x(1-x)(1 + \varepsilon \cos(2\pi\theta)), \end{array}\right\} \tag{2}$$

where (α, ε) are parameters and ω a fixed Diophantine number. The dynamics on the periodic component is a rigid rotation and the dynamics on the real component is the Logistic Map plus a perturbation depending on the periodic one. Sometimes the FLM is also defined with $f_{\alpha,\varepsilon}(\theta, x) = \alpha x(1-x) + \varepsilon \cos(2\pi\theta)$. The results in this paper applies to both cases.

The FLM map appears in the literature in different contexts, usually related with the destruction of invariant curves, see [12] and references therein. Concretely, we are interested on the truncation of the period doubling cascade. As discussed above, the Logistic Map exhibits an infinite cascade of period doubling bifurcations which leads to chaotic behavior. For zero coupling ($\varepsilon = 0$), these periodic orbits become invariant curves of the FLM (provided the rotation number ω is irrational). But when the coupling parameter is different from zero, the number of

period doubling bifurcations of the invariant curves is finite.

We studied numerically this phenomenon in [12]. Concretely, we computed some bifurcation diagrams in terms of the dynamics of the attracting set, taking into account different properties, as the Lyapunov exponent and, in the case of having a periodic invariant curve, its period and its reducibility. In the case of analytic maps in the cylinder, the reducibility loss of an invariant curve can be characterized as a bifurcation (see definition 2.3 in [12]). But the reducibility loss is not a bifurcation in the classical sense because there are not visible changes in the phase space, only the spectral properties of the transfer operator associated to the continuation of that curve changes (see [13]). Despite of this, we will consider it as a bifurcation for the rest of this paper. The numerical computations in the cited work reveal that the parameter values for which the invariant curve doubles its period are contained in regions of the parameter space where the invariant curve is reducible. As before, let s_n be the parameter values where the critical point of the uncoupled family is periodic with period 2^n. The numerical computations also revealed that from every parameter value $(\alpha, \varepsilon) = (s_n, 0)$ two curves are born. These curves correspond to a reducibility-loss bifurcation of the 2^n-periodic invariant curve. The scenario is sketched in figure 1.

Assume that these two curves can be locally expressed as $(\alpha_n^+(\varepsilon), \varepsilon)$ and $(\alpha_n^-(\varepsilon), \varepsilon)$ with $\alpha_n^+(0) = \alpha_n^-(0) = s_n$. In theorem 3.8 we prove that these curves really exist for suitable families of maps. Moreover, we give the values of $\frac{d}{d\varepsilon}\alpha_n^+(0)$ and $\frac{d}{d\varepsilon}\alpha_n^-(0)$ in terms of the iterates of the renormalization operator. To achieve this result we need to assume that the quasi-periodic renormalization operator is injective. This assumption will be called Conjecture **A** which is stated in section 3.1. This conjecture will be supported numerically in [23].

The paper is structured as follows. In section 2 we propose a definition for the q.p. renormalization operator and we study the operator as a map on the Banach space of q.p. forced unimodal maps. Among other results, we prove that the fixed point of the one dimensional renormalization operator extends to the quasi-periodic one and we compute and study its derivative. In section 3 we consider certain codimension one manifolds, which correspond to the bifurcation manifold associated to the reducibility loss of the 2^n periodic invariant curve of the system. We relate these manifolds for different values of n by means of the renormalization operator. Then we consider a generic two parametric family such that it becomes a full family of renormalizable one dimensional maps when one of its parameters is equal to zero. We use the q.p. renormalization theory to prove the existence of reducibility loss bifurcations for the family. We also include an appendix where we analyze the minimum function as a functional operator from the space of functions $f : \mathbb{T} \to \mathbb{R}$ to $\mathbb{R}$, which is necessary for our discussion.

2 Definition of the operator and basic properties

Consider a q. p. forced map as follows,

$$\begin{array}{cccc} F: & \mathbb{T} \times I & \to & \mathbb{T} \times I \\ & \left(\begin{array}{c} \theta \\ x \end{array}\right) & \mapsto & \left(\begin{array}{c} \theta + \omega \\ f(\theta, x) \end{array}\right), \end{array} \tag{3}$$

where $I = [-1, 1]$ and $f \in C^r(\mathbb{T} \times I, I)$. To define the renormalization operator we only require $r \geq 1$, but in this work we focus on the simplest case of analytic functions.

Let us remark that, in this section, no additional assumptions will be done on ω. The aim of this section is to define the quasi-periodic renormalization operator. As long as the dynamics of the

map F is not considered, it is not necessary any additional requirement on ω. In section 3 we will require ω to be Diophantine. But for this section it is advantageous to define the operator for any $\omega \in \mathbb{T}$, since then the operator will depend continuously on ω.

The definition of the renormalization operator will be done from a perturbative point of view. In other words, we will consider a map F like (3) such that $f(\theta, x) = f_0(x) + h(\theta, x)$, with f_0 a unimodal map on $\mathcal{D}(\mathcal{R})$ the domain of the renormalization operator. In this section we will see that if h small enough (in $\|\cdot\|_\infty$ norm), then we can define a "renormalization" of f.

In section 2.1 we introduce the setup of the one dimensional renormalization operator that we consider in this paper. In section 2.2 we define the renormalization operator for q.p. forced maps and we will check that the definition is consistent. In section 2.3 the basic properties of the operator are studied. Concretely, we check that a fixed point of the one dimension renormalization operator extends to a fixed point of the q.p. one and we will also check that the operator is differentiable in a neighborhood of the fixed point. In subsection 2.4 we study the derivative of the operator with respect to the Fourier expansion of the function to which the operator is applied.

2.1 Setup of the one dimensional renormalization operator.

We introduce here the precise definition of the one dimensional renormalization operator. The approach chosen here is due to its simplicity, which makes easier to adapt to the quasi periodic case. Concretely we follow [17], but we do a slight modification on the domain of the operator for technical reasons. For a set up in a much more general context see [24, 4, 19] and for more recent works on renormalization of one-dimensional maps see [3] and [2].

For the understanding of this section it is advisable to have some familiarity with the definition of the (doubling) renormalization operator given in [17]. Note that the definition given there is for even maps defined on the interval $[-1, 1]$, such that the turning point is 0 and it is mapped to 1. Given a skew map F like (3), we want to define the renormalization of the map in a similar way to the one-dimensional case. That is, to give some generic conditions on F in such a way that it has a two periodic invariant subset, and such that F^2 restricted to this subset is affinely conjugate to a map in the same class of functions of F. Note that the θ-component of F, when ω is irrational, does not allow the map to have invariant subsets in this component. Moreover we want the renormalization of F to have a rigid rotation in the periodic component. Then the affine conjugacy should be of the form $A(\theta, x) = (\theta, ax)$, with a a real number. Note that the skew map $A^{-1} \circ F \circ F \circ A$ has rotation number equal to 2ω and is defined by the function $\frac{1}{a} f(\theta + \omega, f(\theta, ax))$,

Suppose that we have g a renormalizable one dimensional map and $h \in C^r(\mathbb{T} \times I, I)$ such that its supremum norm $\|h\|$ is small. Then we would like to consider a map F like (3) defined by the function

$$f(\theta, x) = g(x) + h(\theta, x).$$

Note that the definition of the renormalization operator given in [17] is for even maps defined on the interval $[-1, 1]$, such that the turning point is 0 and it is mapped to 1. If we want $F : \mathbb{T} \times I \to \mathbb{T} \times I$ to be well defined we should have $f(\mathbb{T} \times I) \subset I$. Although we allow $\|h\|$ to be small, we should require $h(\theta, 0)$ to be negative for any $\theta \in \mathbb{T}$. This would make the construction quite artificial and not applicable to the general q.p. forced maps like the FLM. A solution to this problem is to replace the interval $I = [-1, 1]$ by a wider one $I_\delta = [-1-\delta, 1+\delta]$, but then we

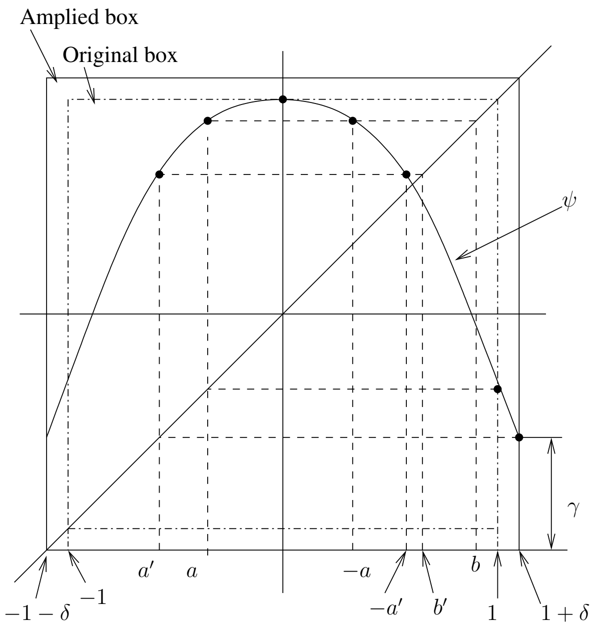

Figure 2: Given a function $\psi \in \mathcal{M}_\delta$, in the picture we show the geometrical meaning of the new constants a', b', δ and γ.

have to check that the one dimensional renormalization operator can be extended to this new domain.

Given a small value δ, let $\mathcal{M}_\delta$ denote the space of continuously differentiable even maps ψ of the interval $I_\delta = [-1-\delta, 1+\delta]$ into itself such that

1. $\psi(0) = 1$,
2. $x\psi'(x) < 0$ for $x \neq 0$.

Set $a = \psi(1)$, $a' = (1+\delta)a$ and $b' = \psi(a')$. In figure 2 one can see an example of a function in $\mathcal{M}_\delta$ where these values are shown. Now, we define $\mathcal{D}(\mathcal{R}_\delta)$ as the set of $\psi \in \mathcal{M}_\delta$ such that

1. $a < 0$
2. $1 > b' > -a'$,
3. $\psi(b') < -a'$.

Remark 2.1. Note that the values a, a' and b' can be seen as continuous functions from $\mathcal{M}_\delta$ to $\mathbb{R}$, and therefore the set $\mathcal{D}(\mathcal{R}_\delta)$ is open in $\mathcal{M}_\delta$ (with the C^k topology).

We define the renormalization operator, $\mathcal{R}_\delta : \mathcal{D}(\mathcal{R}_\delta) \to \mathcal{M}_\delta$ as

$$\mathcal{R}_\delta(\psi)(x) = \frac{1}{a}\psi \circ \psi(ax). \tag{4}$$

where $a = \psi(1)$.

Proposition 2.2. *The operator $\mathcal{R}_\delta$ is well defined, in the sense that $\mathcal{R}_\delta(\psi)$ belongs to $\mathcal{M}_\delta$ for any $\psi \in \mathcal{D}(\mathcal{R}_\delta)$*

Proposition 2.3. *Any fixed point $\Phi \in \mathcal{M}_0$ of $\mathcal{R}_0$ extends to a fixed point of $\mathcal{R}_\delta$, as long as $\Phi \in \mathcal{D}(\mathcal{R}_\delta)$.*

Let us remark that in the proof of the existence of a fixed point done in [17] it is claimed that there exists a function Φ, which is analytic on the domain $\left\{z \in \mathbb{C} \middle| |z^2 - 1| \leq \sqrt{8}\right\}$, and such that its restriction to $I = [-1, 1]$ is a fixed point of $\mathcal{R}$. Then the fixed point Φ extends to a fixed point of $\mathcal{R}_\delta$ as long as $1 + \delta < \sqrt{8}$ and $\Phi(1 + \delta) > -(1 + \delta)$. Concretely we have that there exist δ_0 (small enough) such that Φ extends to a fixed point of $\mathcal{R}_\delta$ for any $0 < \delta < \delta_0$. For the rest of this paper δ is fixed equal to δ_0.

Proofs

Proof of proposition 2.2. Given a function $\psi \in \mathcal{D}(\mathcal{R}_\delta)$, let $\Psi = \mathcal{R}_\delta(\psi)$. Then

$$\Psi(0) = \frac{1}{a}\psi \circ \psi(0) = \frac{1}{a}\psi(1) = 1.$$

It is easy to check that

$$x\Psi'(x) = x\psi'(ax)\psi'(\psi(ax)) \tag{5}$$

Then, using that $x\psi'(x) < 0$, for any $x \in I_\delta$ and $x \neq 0$, we have that $x\psi'(-ax) \geq 0$, for any $x \in I_\delta$ and $x \neq 0$. On the other hand, for any $x \in I_\delta$ we have that $\psi(-ax) \in [b', 1]$. Using again $x\psi'(x) < 0$ and $0 < a < a' < b'$ we have that $\psi'(\psi(ax)) < 0$ for any $x \in I_\delta$. Then we can conclude that $x\Psi'(x) < 0$ for any $x \in I_\delta$ and $x \neq 0$.

The last condition to check is that for any $x \in I_\delta$, the map Ψ maps x inside the set I_δ. Using that $\Psi(0) = 1$ and the monotonicity consequences of $x\Psi'(x) < 0$ we have that $\Psi(x) < \Psi(0) = 1$. We only have to check that $\Psi(1 + \delta) > -1 - \delta$. Since $\Psi(1 + \delta) = -\frac{1}{a}\psi(b')$, then the inequality holds from $\psi(b') < a'$. □

Proof of proposition 2.3. A fixed point of the renormalization operator can be extended to the real line using recursively the invariance equation $\psi(x) = \frac{1}{a}\psi \circ \psi(ax)$ to evaluate points from outside of I (recall that $|a| < 1$). Moreover we have to check that $\psi \in \mathcal{D}(\mathcal{R}_\delta)$. Using again that $|a| < 1$, we have that $a = f(1) > -1$, therefore for a sufficiently small δ we will have $f(1 + \delta) > -1 - \delta$. □

2.2 The renormalization operator for quasi-periodically forced maps

In this section we define the quasi-periodic renormalization operator and we check that there exists a non-empty set of maps where it is well defined.

Consider the operator

$$\begin{array}{rccc} p_0 : & C^r(\mathbb{T} \times I_\delta, I_\delta) & \rightarrow & C^r(I_\delta, I_\delta) \\ & f(\theta, x) & \mapsto & \displaystyle\int_0^1 f(\theta, x) d\theta. \end{array} \tag{6}$$

If we consider the natural inclusion of $C^r(I_\delta, I_\delta)$ into $C^r(\mathbb{T} \times I_\delta, I_\delta)$ then we have that p_0 is a projection ($(p_0)^2 = p_0$).

Consider $\mathcal{M}_\delta \subset C^r(I_\delta, I_\delta)$ defined in the previous subsection. Then we can consider the space $\mathcal{X}$ defined as:

$$\mathcal{X} = \{f \in C^r(\mathbb{T} \times I_\delta, I_\delta) | \, p_0(f) \in \mathcal{M}_\delta\},$$

and the decomposition $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_0^c$ of it given by the projection p_0, i.e $\mathcal{X}_0 = \{f \in \mathcal{X} \mid p_0(f) = f\}$ and $\mathcal{X}_0^c = \{f \in \mathcal{X} \mid p_0(f) = 0\}$. Note that from the definition of $\mathcal{X}$ it follows that $\mathcal{X}_0$ is an isomorphic copy of $\mathcal{M}_\delta$.

Proposition 2.4. *Let f be a function in $\mathcal{X}_0$ and consider $\gamma := f(1+\delta) + 1 + \delta$ (see figure 2). Consider also $h(x, \theta) \in C^r(\mathbb{T} \times I_\delta, I_\delta)$ with $p_0(h) = 0$. If $\|h\| < \delta$ and $\|h\| < \gamma$, then $g = f + h$ belongs to $\mathcal{X}$.*

In other words, we have that small quasi-periodic perturbations of an uncoupled one dimensional map of $\mathcal{M}_\delta$ belong to the space considered here. With this space of functions we are able to define the q.p. renormalization operator. The proof of proposition 2.4 is at the end of the subsection.

Definition 2.5. Given a function $g \in \mathcal{X}$, we can define the **renormalization** of g as

$$[\mathcal{T}_\omega(g)](\theta, x) := \frac{1}{\hat{a}} g(\theta + \omega, g(\theta, \hat{a}x)), \tag{7}$$

where $\hat{a} = \displaystyle\int_0^1 g(\theta, 1) d\theta$. Consider the set $\mathcal{D}(\mathcal{T}) = \{g \in \mathcal{X} \mid \mathcal{T}_\omega(g) \in \mathcal{X}\}$, then the renormalization operator $\mathcal{T}_\omega$ is defined from $\mathcal{D}(\mathcal{T})$ to $\mathcal{X}$.

Remark 2.6. The choice of $\hat{a} = \int_0^1 g(\theta, 1) d\theta$ is somehow arbitrary. In a more general context, one might allow $\hat{a}$ to be indeed a function of the angle, but the first problem arises with the choice of this function. We have chosen a to be constant in order to keep the problem as simple as possible. For our purposes we find this approach sufficient, since we are always considering maps which are perturbations of one dimensional maps.

Note that, with the definition given above, it might happen that $D(\mathcal{T}) = \emptyset$. Then let us ensure that this is not the case.

The space $\mathcal{X}_0$, which is an isomorphic copy $\mathcal{M}_\delta$, is a subspace of $\mathcal{X}$. Then let us define $\mathcal{D}_0(\mathcal{T})$ as the set of functions in $\mathcal{X}_0$ that correspond to functions of $\mathcal{D}(\mathcal{R}_\delta) \subset \mathcal{M}_\delta$. For any function $g \in \mathcal{D}_0(\mathcal{T})$ we have $\mathcal{T}_\omega(g) = \mathcal{R}_\delta(g) \in \mathcal{X}$, therefore $\mathcal{D}_0(\mathcal{T})$ is a subset of $\mathcal{D}(\mathcal{T})$ and consequently $\mathcal{D}(\mathcal{T})$ is not empty. Indeed we have the following result on the topology of $\mathcal{D}(\mathcal{T})$.

Proposition 2.7. *There exists an open set W in $\mathcal{X}$ such that $\mathcal{T}_\omega : W \to C^r(\mathbb{T} \times I_\delta, I_\delta)$ is a well defined continuous map. Additionally consider $U = (p_0 \circ \mathcal{T}_\omega)^{-1}(\mathcal{M}_\delta)$, then we have that there exists an open set W' in $U \cap \mathcal{X}$ such that $\mathcal{D}_0(\mathcal{T}) \subset W' \subset \mathcal{D}(\mathcal{T})$.*

Remark 2.8. Note that the sets $\mathcal{X}$, $\mathcal{D}(\mathcal{T})$, $\mathcal{D}_0(\mathcal{T})$ and the operator $\mathcal{T}_\omega$ depend on the value δ, but here we do not make it explicit to keep the notation simple. On the other hand, we made explicit the dependence of $\mathcal{T}_\omega$ on ω, but not in the set $\mathcal{D}(\mathcal{T})$ and $\mathcal{D}_0(\mathcal{T})$, which a priori should also depend on ω. The set $\mathcal{D}_0(\mathcal{T})$ is an isomorphic copy of $\mathcal{D}(\mathcal{R}_\delta)$, then it does not depend on ω. The set $\mathcal{D}(\mathcal{T})$ actually depends on ω, but it is omitted from the notation for simplicity.

Remark 2.9. Consider a map F like (3). We have that the map is determined by a function $f \in \mathcal{X}$ and a value of $\omega \in \mathbb{T}$. Then we can define the renormalization of (ω, f) as the map determined by $(2\omega, \mathcal{T}_\omega f)$, which acts from $\mathbb{T} \times \mathcal{D}(\mathcal{T})$ to $\mathbb{T} \times \mathcal{X}$. Note that the frequency ω has been doubled. This is due to the fact that the renormalization of a map is constructed as the affine transformation of the map iterated twice. For convenience, in the remaining of this section we will study $\mathcal{T}_\omega$ as an operator acting from $\mathcal{D}(\mathcal{T})$ to $\mathcal{X}$ depending on the parameter ω. In section 3 we will take into account the doubling of the rotation number again.

Proofs

Proof of proposition 2.4. Note that $p_0(g) = p_0(f) + p_0(h) = f \in \mathcal{M}_\delta$. To prove that $g \in \mathcal{X}$, it is only necessary to check that $g \in C^r(\mathbb{T} \times I_\delta, I_\delta)$. The map g is C^r for being the linear combination of C^r maps. The function g belongs to $C^r(\mathbb{T} \times I_\delta, I_\delta)$ if $g(\theta, x) \in I_\delta$ for any $\theta \in \mathbb{T}$ and $x \in I_\delta$.

We begin with the upper bound. As $f(x) \leq f(0) = 1$ for any $x \in I_\delta$, then

$$g(\theta, x) = f(x) + h(\theta, x) \leq f(x) + \|h\| \leq f(0) + \varepsilon = 1 + \varepsilon,$$

for any $(\theta, x) \in \mathbb{T} \times I_\delta$.

Now we check the lower bound. We have that γ only depends on f and it is always greater or equal to 0. On the other hand, we have $f(x) \geq f(1+\delta)$ for any $x \in I_\delta$. Then

$$g(\theta, x) = f(x) + h(\theta, x) \geq f(x) - \|h\| \geq f(1+\delta) - \|h\| = -1 - \delta + \gamma - \|h\|,$$

which implies that $g(\theta, x) \geq -1 - \delta$ if $\|h\| < \gamma$. □

Proof of proposition 2.7. First of all we need to build an open set where the q.p. renormalization operator (7) is well defined. With this aim we have the following lemma.

Lemma 2.10. *There exists an open set $U_1 \subset \mathcal{X}$ with $\mathcal{D}_0(\mathcal{T}) \subset U_1$ such that the map*

$$\begin{array}{rccc} F_1: & U_1 \subset C^r(\mathbb{T} \times I_\delta, I_\delta) & \to & C^r(\mathbb{T} \times I_\delta, I_\delta) \\ & g = g(\theta, x) & \mapsto & [F_1(g)](\theta, x) = g(\theta, \hat{a}x), \end{array}$$

with $\hat{a} = \displaystyle\int_0^1 g(\theta, 1)d\theta$, is well defined and continuous.

Proof. Given a function $g \in \mathcal{X}$ we have that $p_0(g) \in \mathcal{M}_\delta$. Note that the value $\hat{a}$ as a functional operator $\hat{a} : \mathbb{C}^r(\mathbb{T} \times I_\delta, I_\delta) \to I_\delta$ is equal to the evaluation map at $x = 1$ composed with the projection p_0. This is a bounded linear operator, therefore we have that $\hat{a} : C^r(\mathbb{T} \times I_\delta, I_\delta) \to I_\delta$ is continuous. Then we can consider $U_1 = \hat{a}^{-1}((-1, 1))$, which is an open set because it is the preimage of an open set by a continuous function. For any $g \in U_1$ using $|\hat{a}| < 1$ we have

$$\sup_{(\theta,x)\in\mathbb{T}\times I_\delta} |g(\theta, \hat{a}x)| \leq \sup_{(\theta,x)\in\mathbb{T}\times I_\delta} |g(\theta, x)| \leq 1 + \delta.$$

Hence $g(\theta, \hat{a}x)$ is well defined for any $g \in U_1$ and $(\theta, x) \in \mathbb{T} \times I_\delta$. Moreover for any $g \in \mathcal{D}_0(\mathcal{T})$ we have $\hat{a}(g) = [p_0(g)](1) \in (-1, 0)$, which proves that $\mathcal{D}_0(\mathcal{T}) \subset U_1$. □

As discussed in the proof of lemma 2.10 above we have that $\hat{a} : U_1 \to (-1, 1)$ defined as $\hat{a}(g) = \int_0^1 g(\theta, 1)d\theta$ is a continuous function.

Using the results on the smoothness of the composition map from [11] and lemma 2.10 above, we have that the map

$$\begin{array}{rccc} F_2: & U_1 \subset C^r(\mathbb{T} \times I_\delta, I_\delta) & \to & C^r(\mathbb{T} \times I_\delta, I_\delta) \\ & g = g(x, \theta) & \mapsto & [F_2(g)](\theta, x) = g(\theta + \omega, g(\theta, \hat{a}x)), \end{array}$$

is well defined and is also continuous.

On the other hand, let $U_2 = p_0^{-1}(\mathcal{D}_0(\mathcal{T}))$ and consider

$$\begin{array}{rccc} F_3: & U_2 \subset C^r(\mathbb{T}\times I_\delta, I_\delta) & \rightarrow & C^r(\mathbb{T},\mathbb{R}) \\ & g = g(x,\theta) & \mapsto & [F_3(g)](x) = \dfrac{1}{\hat{a}}x. \end{array}$$

For any $g \in U_2$ we have $\hat{a}(g) < 0$, therefore the map F_3 is well defined. Indeed, we have that it is continuous with respect to g.

Finally note that $\mathcal{T}_\omega$ is obtained as the composition $(F_3(g)) \circ (F_2(g))$. Using the results from [11] we have that $\mathcal{T}_\omega$ is continuous (and it is well defined) as an operator $\mathcal{T}_\omega : U_3 \subset C^r(\mathbb{T}\times I_\delta, I_\delta) \rightarrow C^r(\mathbb{T}\times I_\delta, \mathbb{R})$, where $U_3 = U_2 \cap U_1$. Note that $\mathcal{D}_0(\mathcal{T}_\omega)$ is the image by the inclusion of $\mathcal{D}(\mathcal{R}_\delta)$. Therefore we have $\mathcal{D}_0(\mathcal{T}_\omega) \subset U_2$ and, consequently, $\mathcal{D}_0(\mathcal{T}_\omega) \subset U_3$.

Note that the image space of the operator is $\mathcal{T}_\omega$ is not the desired one. Since $\mathbb{T}\times I_\delta$ is compact, we have that the map $N : C^r(\mathbb{T}\times I_\delta, \mathbb{R}) \rightarrow [0,\infty)$ defined as $N(g) = \|g\|_\infty$ is continuous. Consider the set $U_4 := N^{-1}([0, 1+\delta))$ which is an open subset of $C^r(\mathbb{T}\times I_\delta, \mathbb{R})$. At this point we can define the set W in the statement of the proposition as

$$W = \mathcal{T}_\omega^{-1}(U_4) \cap U_3.$$

Using this construction of the set W we have $\mathcal{T}_\omega : W \subset C^r(\mathbb{T}\times I_\delta, I_\delta) \rightarrow C^r(\mathbb{T}\times I_\delta, I_\delta)$. Again we have $\mathcal{D}_0(\mathcal{T}_\omega) \subset W$ due to the fact that for any $g \in \mathcal{D}_0(\mathcal{T}_\omega)$ we have $\|\mathcal{T}_\omega(g)\|_\infty = \|\mathcal{R}_\delta(g)\|_\infty < 1+\delta$. This concludes the proof of the first assertion in the proposition.

We check now the second assertion of the proposition. Consider $U = (p_0 \circ \mathcal{T}_\omega)^{-1}(\mathcal{M}_\delta)$ and $U_5 = W \cap U$. Consider also the following auxiliary functions,

$$\begin{array}{rccc} F_4: & U_5 \subset C^r(\mathbb{T}\times I_\delta, I_\delta) & \rightarrow & [0,+\infty), \\ & g = g(x,\theta) & \mapsto & \|\mathcal{T}_\omega(g) - p_0(\mathcal{T}_\omega(g))\|. \end{array}$$

and

$$\begin{array}{rccc} F_5: & U_5 \subset C^r(\mathbb{T}\times I_\delta, I_\delta) & \rightarrow & \mathbb{R}, \\ & g = g(x,\theta) & \mapsto & 1+\delta+[\mathcal{T}_\omega(g)](1+\delta) - \|\mathcal{T}_\omega(g) - p_0(\mathcal{T}_\omega(g))\|). \end{array}$$

Now we can define the set W' in the statement of the proposition as

$$W' := U_5 \cap F_4^{-1}([0,\delta)) \cap F_5^{-1}((-\delta,\delta)).$$

First let us check that $\mathcal{D}_0(\mathcal{T}_\omega) \subset W \subset \mathcal{D}(\mathcal{T})$. For any map $g \in \mathcal{D}_0(\mathcal{T}_\omega)$ we have that $\mathcal{T}_\omega(g) = \mathcal{R}_\delta(g) = p_0(\mathcal{R}_\delta(g)) = p_0(\mathcal{T}_\omega(g))$; then it follows easily that $\mathcal{D}_0(\mathcal{T}_\omega) \subset F_4^{-1}([0,\delta))$. Moreover as $\mathcal{T}_\omega(g) = \mathcal{R}_\delta(g) \in \mathcal{M}_\delta$, we have $[\mathcal{R}_\delta(g)](1+\delta) < 1+\delta$, which implies that $g \in F_5^{-1}(-\delta,\delta)$.

Finally, we have to check $W \subset \mathcal{D}(\mathcal{T})$, which is equivalent to prove that $\mathcal{T}_\omega(g) \in \mathcal{X}$ for any $g \in W$. Given $g \in W$, we have $\mathcal{T}_\omega(g) = p_0(\mathcal{T}_\omega(g)) + \mathcal{T}_\omega(g) - p_0(\mathcal{T}_\omega(g))$. From $g \in U_5$ we have that $p_0(\mathcal{T}_\omega(g)) \in \mathcal{M}_\delta$. Moreover from $g \in F_4^{-1}([0,\delta))$ we have $\|\mathcal{T}_\omega(g) - p_0(\mathcal{T}_\omega(g))\| < \delta$ and from $g \in F_5^{-1}([0,\delta))$ we have $\|\mathcal{T}_\omega(g) - p_0(\mathcal{T}_\omega(g))\| < 1+\delta+[p_0(\mathcal{T}_\omega)](1+\delta)$. Note that as $p_0(\mathcal{T}_\omega(g) - p_0(\mathcal{T}_\omega(g))) = 0$, we can apply proposition 2.4 and then it follows that $g \in \mathcal{X}$. $\square$

2.3 Study of the operator $\mathcal{T}_\omega$

Let us follow with the study of the operator $\mathcal{T}_\omega$. In this section we start showing that the fixed points of $\mathcal{R}_\delta$ extend to fixed points of $\mathcal{T}_\omega$. Then we give a result on the differentiability of $\mathcal{T}_\omega$, in the C^r topology. With this result it becomes evident that the C^r topology is a bad choice for the study of the operator. Lastly, we introduce the topology of real analytic maps and we check that the operator is well defined and differentiable if certain hypothesis (which will be called **H0**) is satisfied. Again, all the proofs have been moved to the end of the section.

Proposition 2.11. *The operator $\mathcal{T}_\omega$ restricted to the set $\mathcal{D}_0(\mathcal{T})$, is isomorphically conjugate to $\mathcal{R}_\delta$. Concretely we have that any fixed point of $\mathcal{R}_\delta$ extends to a fixed point of $\mathcal{T}_\omega$.*

We have the following result on the differentiability of $\mathcal{T}_\omega$.

Theorem 2.12. *Let $\mathcal{T}_\omega : \mathcal{D}(\mathcal{T}) \to \mathcal{X}$ be the renormalization operator in the C^r-topology, and consider Φ a fixed point of the operator. If $\Phi \in \mathcal{D}_0(\mathcal{T}) \cap C^{r+s}(\mathbb{T} \times I_\delta, I_\delta)$ then we have that there exists U an open neighborhood of Φ in $\mathcal{D}_0(\mathcal{T}) \cap C^{r+s}(\mathbb{T} \times I_\delta, I_\delta)$ such that $\mathcal{T}_\omega$ is a C^s operator in U. Moreover, if $s \geq 1$ for any point $\Psi \in U$ we have that the Gateaux derivative of $\mathcal{T}_\omega$ on Ψ in the direction h is given by*

$$\begin{aligned}[][d\mathcal{T}_\omega(\Psi, h)](\theta, x) = \quad & \frac{1}{a}\psi'(\psi(ax))h(\theta, ax) + \frac{1}{a}h(\theta + \omega, \psi(ax)) \\ & + \frac{b}{a}\psi'(\psi(ax))\psi'(ax)x - \frac{b}{a^2}\psi(\psi(ax)),\end{aligned} \tag{8}$$

where $\psi = p_0(\Psi)$, $a = \psi(1)$ and $b = \int_0^1 h(\theta, 1)d\theta$.

Note that there is a "loss of differentiability", in the sense that one needs to assume that the function Ψ where we differentiate the operator is in C^{r+s} while the operator acts in subsets of $C^r(\mathbb{T} \times I_\delta, I_\delta)$. This is due to the self composition in the renormalization operation. To skip this problem, let us introduce the topology of analytic functions instead of the C^r one, for the forthcoming study of the operator.

Definition 2.13. Let $\mathbb{W}$ be an open set in the complex plane containing the interval I_δ and let $\mathbb{B}_\rho = \{z = x + iy \in \mathbb{C} \text{ such that } |y| < \rho\}$. The we define the set $\mathcal{B} = \mathcal{B}(\mathbb{B}_\rho, \mathbb{W})$ as the space of functions $f : \mathbb{B}_\rho \times \mathbb{W} \to \mathbb{C}$ such that:

1. f is holomorphic in $\mathbb{B}_\rho \times \mathbb{W}$ and continuous in the closure of $\mathbb{B}_\rho \times \mathbb{W}$.
2. f is real analytic (it maps real numbers to real numbers).
3. f is 1-periodic in the first variable, i. e. $f(\theta + 1, z) = f(\theta, z)$ for any $(\theta, z) \in \mathbb{B}_\rho \times \mathbb{W}$.

Proposition 2.14. *The space $\mathcal{B}$ endowed with the supremum norm $\left(\|f\| = \sup_{\mathbb{B}_{\rho_1} \times \mathbb{W}} |f(\theta, z)| \right)$ is a Banach space.*

We want to consider the quasi-periodic renormalization operator $\mathcal{T}_\omega$ (see equation (7)) restricted to the domain $\mathcal{D}(\mathcal{T}) \cap \mathcal{B}$, but then it is necessary to check that $\mathcal{T}_\omega$ is well defined in the complex domain. For any function $f \in \mathcal{D}(\mathcal{T}) \cap \mathcal{B}$ we should check that $f(\mathbb{B}_\rho \times a\mathbb{W}) \subset \mathbb{W}$ (where $a\mathbb{W} = \{z \in \mathbb{C} |\, az \in \mathbb{W}\}$). In the one dimensional renormalization theory the open set $\mathbb{W}$ is

chosen such that this condition is satisfied. Concretely, this condition is typically checked with computer assistance, together with other conditions to prove the existence of fixed points (see [17, 18, 15]). In our case we will assume that the following hypothesis is satisfied.

H0) There exists an open set $\mathbb{W} \subset \mathbb{C}$ containing I_δ and a function $\Phi \in \mathcal{B} \cap \mathcal{X}_0$ such that $p_0(\Phi)$ is a fixed point of the renormalization operator $\mathcal{R}_\delta$ and such that the closure of both $a\mathbb{W}$ and $p_0(\Phi)(a\mathbb{W})$ is contained in $\mathbb{W}$ (where $a := \Phi(1)$)

Although the results on the existence of the fixed point of renormalization operator done by Lanford in [17] are well accepted, in the cited article there are no proofs and many details are omitted. For the proof of the existence of the fixed point it is also necessary to check this condition. In [18], some more details on how to prove the existence of the fixed point are given. Actually, it is claimed that the hypothesis **H0** is true for the set

$$\left\{ z \in \mathbb{C} \text{ such that } |z^2 - 1| < \frac{5}{2} \right\}.$$

This set used by Lanford is more convenient in his study since he works in the set of even holomorphic functions. In the numerical computations from [23] we use as $\mathbb{W}$ the disc centered at $\frac{1}{5}$ with radius $\frac{3}{2}$, and we check hypothesis **H0** numerically (without rigorous bounds).

Theorem 2.15. *Assume that* **H0** *holds and let Φ be the fixed point given by this assumption. Then we have that there exists $U \subset \mathcal{D}(\mathcal{T}) \cap \mathcal{B}$, an open neighborhood of Φ, such that $\mathcal{T}_\omega : U \to \mathcal{B}$ is well defined, and $\mathcal{T}_\omega$ is Fréchet differentiable for any $\Psi \in U$ and the derivative is equal to*

$$\begin{aligned}
[D\mathcal{T}_\omega(\Psi, h)](\theta, x) = \quad & \frac{1}{a}(\partial_x \Psi)(\theta + \omega, \Psi(\theta, ax))h(\theta, ax) + \frac{1}{a}h(\theta + \omega, \Psi(\theta, ax)) \\
& + \frac{b}{a}(\partial_x \Psi)(\theta + \omega, \Psi(\theta, ax))(\partial_x \Psi(\theta, ax))x - \frac{b}{a^2}\Psi(\theta + \omega, \Psi(\theta, ax)),
\end{aligned} \tag{9}$$

where $a = \int_0^1 \Psi(\theta, 1) d\theta$ and $b = \int_0^1 h(\theta, 1) d\theta$.

Proofs

Proof of proposition 2.11. If a map belongs to $\mathcal{D}_0(\mathcal{T})$ then it does not depend on θ, therefore the operator $\mathcal{T}_\omega$ coincides with $\mathcal{R}_\delta$ composed with the inclusion of $\mathcal{M}_\delta$ in $\mathcal{X}$. □

Proof of theorem 2.12. By definition (see equation (7)) given a function $g \in C^r(\mathbb{T} \times I_\delta, I_\delta)$ we have that $[\mathcal{T}_\omega(g)](\theta, x) := \frac{1}{\hat{a}} g(\theta + \omega, g(\theta, \hat{a}x))$ where $\hat{a} = \int_0^1 g(\theta, 1) d\theta$. Note that $\hat{a}$ can be written as $\hat{a} = [p_0(g)](1)$. The function $p_0 : C^{r+s}(\mathbb{T} \times I_\delta, I_\delta) \to C^{r+s}(I_\delta, I_\delta)$ defined by (6) is C^s (actually it is a linear bounded operator). On the other hand the evaluation of a C^r function in a concrete value is also a C^r function (see proposition 2.4.17 from [1]). Therefore $\hat{a} = \hat{a}(g)$ as a function of g is C^r as well.

Note that $\mathcal{T}_\omega(g)$ can be written as the composition of different functions, which are g itself, a translation in the θ variable and a scalar multiplication by a (and its inverse) in the x variable. Each one of these functions are C^r dependent with respect to f except the composition of f

with itself which is only a C^s map, when we work in the C^r topology (see [11]). We can conclude that $\mathcal{T}_\omega$ is only a C^s operator.

Now we compute explicitly the Gateaux derivative. As Ψ belongs to $\mathcal{X}_0$, we have $\Psi(x,\theta) = \psi(x)$ and consequently

$$\begin{aligned}
\left[\mathcal{T}(\Psi + th)\right](\theta, x) &= \frac{1}{\hat{a}}\left[\Psi + th\right](\theta + \omega, \left[\Psi + th\right](\theta, \hat{a}x)) \\
&= \frac{1}{\hat{a}}\psi(\psi(\hat{a}x)) + \frac{1}{\hat{a}}\psi'(\psi(\hat{a}x))h(\theta, \hat{a}x)t \\
&\quad + \frac{1}{\hat{a}}h(\theta + \omega, \psi(\hat{a}x))t + O(t^2),
\end{aligned} \tag{10}$$

where $\hat{a} = \displaystyle\int_0^1 [\Psi + th](1,\theta)d\theta$.

Set $a = \psi(1)$ and $b = \displaystyle\int_0^1 h(1,\theta)d\theta$. We have that $\hat{a} = a + tb$. Therefore,

$$\frac{1}{\hat{a}} = \frac{1}{a + tb} = \frac{1}{a} - \frac{1}{a^2}tb + O(t^2), \tag{11}$$

and using the chain rule we have

$$\psi\left(\psi(\hat{a}x)\right) = \psi\left(\psi(ax)\right) + \psi'\left(\psi(ax)\right)\psi'(ax)tbx + O(t^2). \tag{12}$$

Combining equations (10), (11) and (12) follows that the Gateaux derivate of $\mathcal{T}$ in Ψ is the one given by (9). $\square$

Proof of proposition 2.14. Consider $\mathcal{A}$ the space of holomorphic functions in $\mathbb{B}_\rho \times \mathbb{W}$ and continuous in the closure of $\mathbb{B}_\rho \times \mathbb{W}$. Using basic properties of the holomorphic functions in several variables (see [10, 16]) is easy to check that $\mathcal{A}$ is a Banach space. The space $\mathcal{B}$ is the set of functions of $\mathcal{A}$ such that

- $f(\theta + 1, z) - f(\theta, z) = 0$ for any (θ, z) in $\mathbb{B}_\rho \times \mathbb{W}$.
- $f(\theta, x) - \overline{f(\theta, x)} = 0$ for any (θ, x) in $\mathbb{R} \times I_\delta$.

Then $\mathcal{B}$ is the preimage of a closed subset by a continuous function, therefore it is closed in $\mathcal{A}$ and consequently it is a Banach space. $\square$

Proof of the theorem 2.15. Given a function $f \in \mathcal{D}(\mathcal{T}) \cap \mathcal{B}$, we have that its image by $\mathcal{T}_\omega(f)$ is given by (7). If we want it to be well defined we must check that $f(\mathbb{B}_\rho \times a\mathbb{W}) \subset \mathbb{W}$ (where $a\mathbb{W} = \{z \in \mathbb{C} |\, az \in \mathbb{W}\}$).

We have that $\phi(\theta, x) = \left[p_0(\phi)\right](x)$ for any $\theta \in \mathbb{B}_\rho$. Using hypothesis **H0** we have that $\mathrm{Cl}\left(\Phi\left(\mathbb{B}_\rho \times a\mathbb{W}\right)\right) \subset \mathbb{W}$, where $\mathrm{Cl}(\cdot)$ denotes the closure of the set. If we consider now a function f in a suitable neighborhood of Φ we have that it still maps $\mathbb{B}_\rho \times a\mathbb{W}$ inside $\mathbb{W}$ (if f is close enough to Φ in the topology of $\mathcal{B}$).

To prove the differentiability of the map we will check it directly from the definition of Fréchet derivative.

From $\mathrm{Cl}\left(\Phi\left(\mathbb{B}_\rho \times a\mathbb{W}\right)\right) \subset \mathbb{W}$, and the fact of $\mathbb{W}$ being bounded it follows that $\mathrm{Cl}\left(\Phi\left(\mathbb{B}_\rho \times a\mathbb{W}\right)\right)$ is compact. Consider the following filtration of sets in the complex plane

$$\mathrm{Cl}\left(\Phi\left(\mathbb{B}_\rho \times a\mathbb{W}\right)\right) = K_0 \subset V_0 \subset K_1 \subset V_1 \subset K_2 \subset V_2 = \mathbb{W},$$

with each K_i compact and each V_i open, for $i = 0, 1, 2$.

Consider now $U_1 \subset U$ the open neighborhood of Φ in $\mathcal{B}$ formed by the $\Psi \in \mathcal{B}$ such that

$$\Psi\left(\mathbb{B}_\rho \times a\mathbb{W}\right) \subset V_0.$$

For any map $\Psi \in U_1$ we have that $\mathrm{Cl}\left(\Psi\left(\mathbb{B}_\rho \times \hat{a}\mathbb{W}\right)\right) \subset K_1$.

On the other hand, from $K_2 \subset \mathbb{W}$ and the fact that K_2 is compact and $\mathbb{W}$ open, it follows that there exist a value $r > 0$ such that for any $x_0 \in K_2$ the ball centered on x_0 with radius r is contained in $\mathbb{W}$. Then for any map $f \in \mathcal{B}$ we have

$$\partial_x f(\theta, x_0) = \frac{1}{2\pi i} \int_{|z - x_0| = r} \frac{f(\theta, z)}{(z - z_0)^2} d\theta.$$

Then it follows easily that, for any $f \in \mathcal{B}$ and $x_0 \in K_2$ we have

$$|\partial_x f(\theta, x_0)| \leq \frac{1}{r} \|f\|_\infty.$$

Modifying the same argument, we can check that

$$|\partial^2_{x^2} f(\theta, x_0)| \leq \frac{2}{r^2} \|f\|_\infty.$$

Note that both bounds are uniform for any map in U_1.

Consider $\Psi \in U_1$, and $h \in \mathcal{B}$ with $\|h\|$ small. We want to compute $\mathcal{T}_\omega(\Psi + h)$ up to $O(\|h\|^2)$. First of all we have,

$$\mathcal{T}_\omega(\Psi + h) = \frac{1}{\hat{a}(\Psi + h)} \left[\begin{array}{c} \Psi(\theta + \omega, \Psi(\theta, \hat{a}(\Psi + h)x) + h(\theta, \hat{a}(\Psi + h)x)) + \\ h(\theta + \omega, \Psi(\theta, \hat{a}(\Psi + h)x) + h(\theta, \hat{a}(\Psi + h)x)) \end{array} \right]. \tag{13}$$

To simplify the notation consider

$$a = \int_0^1 \Psi(\theta, 1) d\theta, \quad b = \int_0^1 h(\theta, 1) d\theta.$$

Then we have $\hat{a}(\Psi + h) = a + b$, and

$$|b| \leq \int_0^1 |h(\theta, 1)| d\theta \leq \|h\|.$$

Since $\Psi \in U_1$ we have that for any h with $\|h\|$ sufficiently small $\Psi + h \in U_1$, therefore we have that $\Psi(\theta, (a + b)x) + h(\theta, (a + b)x) \in V_1$. Using the complex Taylor expansion with respect to x up to second order we have

$$\begin{aligned} \Psi(\theta + \omega, \Psi(\theta, (a + b)x) + h(\theta, (a + b)x)) &= \Psi(\theta + \omega, \Psi(\theta, (a + b)x)) + \\ &\quad (\partial_x \Psi)(\theta + \omega, \Psi(\theta, (a + b)x)) h(\theta, (a + b)x)) + \\ &\quad R_2(\theta, x) \end{aligned} \tag{14}$$

with

$$|R_2(\theta,x)| \leq \frac{1}{r^2}\frac{1}{1-\frac{\|h\|}{r}}\|\Psi\|\|h\|^2 = O(\|h\|^2). \tag{15}$$

Analogously we have

$$\begin{aligned} h(\theta+\omega,\Psi(\theta,(a+b)x)+h(\theta,(a+b)x)) \quad &= \quad h(\theta+\omega,\Psi(\theta,(a+b)x))+ \\ & \quad\quad R_1(\theta,x), \end{aligned} \tag{16}$$

with

$$|R_1(\theta,x)| \leq \frac{1}{r}\frac{1}{1-\frac{\|h\|}{r}}\|h\|\|h\| = O(\|h\|^2). \tag{17}$$

Recall that $|b| = O(\|h\|)$ then applying Taylor expansion and the uniform bound on K_2 it follows easily that

$$\Psi(\theta,(a+b)x) \quad = \quad \Psi(\theta,ax)+(\partial_x\Psi)(\theta,ax)bx+O(\|h\|^2), \tag{18}$$

$$h(\theta,(a+b)x) \quad = \quad h(\theta,ax)+O(\|h\|^2). \tag{19}$$

Using that $\Psi \in U_1$ we have that $\Psi(\theta,ax)+(\partial_x\Psi)(\theta,ax)bx$ belongs to $V_1 \subset K_2$ for $\|h\|$ sufficiently small. Now we can combine this fact with the uniform bound on K_2 and equation (18) and (19) to prove that

$$h(\theta+\omega,\Psi(\theta,(a+b)x)) = h(\theta+\omega,\Psi(\theta,ax))+O(\|h\|^2).$$

Using this together with equations (16) and (17) we obtain

$$h(\theta+\omega,\Psi(\theta,(a+b)x)+h(\theta,(a+b)x)) \quad = \quad h(\theta+\omega,\Psi(\theta,ax))+O(\|h\|^2). \tag{20}$$

With the same argument it follows that

$$(\partial_x\Psi)(\theta+\omega,\Psi(\theta,(a+b)x)) = (\partial_x\Psi)(\theta+\omega,\Psi(\theta,ax))+O(\|h\|),$$

and

$$\begin{aligned} \Psi(\theta+\omega,\Psi(\theta,(a+b)x)) \quad &= \quad \Psi(\theta+\omega,\Psi(\theta,ax))+ \\ & \quad\quad (\partial_x\Psi)(\theta+\omega,\Psi(\theta,ax))(\partial_x\Psi)(\theta,ax)bx+O(\|h\|^2). \end{aligned}$$

Replacing the last two equations in equation (14) and using the bound given by (15) yields to

$$\begin{aligned} \Psi(\theta+\omega,\Psi(\theta,(a+b)x)+h(\theta,(a+b)x)) \quad &= \quad \Psi(\theta+\omega,\Psi(\theta,ax))+ \\ & \quad\quad (\partial_x\Psi)(\theta+\omega,\Psi(\theta,ax))(\partial_x\Psi)(\theta,ax)bx+ \\ & \quad\quad (\partial_x\Psi)(\theta+\omega,\Psi(\theta,ax))h(\theta,(a+b)x))+O(\|h\|^2). \end{aligned} \tag{21}$$

Finally, recall that $|b| = O(\|h\|)$, therefore

$$\frac{1}{a+b} = \frac{1}{a} - \frac{b}{a^2} + O(\|h\|).$$

When we replace this value and the ones of equations (20) and (21) in (13) it follows that

$$\|\mathcal{T}_\omega(\Psi+h) - \mathcal{T}_\omega(\Psi) - D\mathcal{T}_\omega(\Psi)h\| = O\left(\|h\|^2\right),$$

which proves the differentiability of the operator in the analytic context. □

2.4 Fourier expansion of $D\mathcal{T}_\omega(\Psi)$.

Let Ψ be a function as in the hypothesis of theorem 2.15, but additionally assume that $\Psi \in U \cap \mathcal{D}_0(\mathcal{T}_\omega)$. In this section we study $D\mathcal{T}_\omega(\Psi)$, the differential of the quasi-periodic renormalization operator. Concretely, given $f \in \mathcal{B}$ we study the Fourier expansion of $D\mathcal{T}_\omega(\Psi)f$ in terms of the Fourier expansion of f. It will turns out that, fixed a positive integer k, the spaces generated by functions of the type $f(x)\cos(2\pi k\theta) + g(x)\sin(2\pi k\theta)$ (with f and g one dimensional real analytic functions) are invariant by $D\mathcal{T}_\omega(\Psi)$. This allows us to reduce the study of $D\mathcal{T}_\omega(\Psi)$ to a simpler operator $\mathcal{L}_\omega$. We finish giving different spectral properties on the operator $\mathcal{L}_\omega$. As usual the proofs have been moved to the end of the section.

Given a function $f \in \mathcal{B}$ we can consider its complex Fourier expansion in the periodic variable

$$f(\theta, z) = \sum_{k\in\mathbb{Z}} c_k(z) e^{2\pi k\theta i}, \tag{22}$$

where

$$c_k(z) = \int_0^1 f(\theta, z) e^{-2\pi k\theta i} d\theta.$$

We can also consider its real Fourier expansion

$$f(\theta, z) = a_0(z) + \sum_{k>0} a_k(z)\cos(2\pi k\theta) + b_k(z)\sin(2\pi k\theta). \tag{23}$$

Here the coefficients are given as,

$$a_0(z) = \int_0^1 f(\theta, z) d\theta,$$

$$a_k(z) = \frac{1}{2}\int_0^1 f(\theta, z)\cos(2\pi k\theta) d\theta, \quad k > 0,$$

$$b_k(z) = \frac{1}{2}\int_0^1 f(\theta, z)\sin(2\pi k\theta) d\theta, \quad k > 0.$$

We have the well known relation between both expansions, given by $c_k(z) = \dfrac{a_k(z) + ib_k(z)}{2}$ when $k > 0$ and $c_0(z) = a_0(z)$.

Note that each function c_k is holomorphic in $\mathbb{W}$ but not real holomorphic (the image of a real number will not be a real number in general). On the other hand the real Fourier coefficient $a_k(z)$ and $b_k(z)$ are real holomorphic in $\mathbb{W}$.

Let $\psi = p_0(\Psi)$ be representative of Ψ as a one dimensional map. Evaluating (9) on $c(z)e^{2\pi k\theta i}$ (for any $k \neq 0$) we have

$$[D\mathcal{T}_\omega(\Psi)]\left(c_k(z)e^{2\pi k\theta i}\right) = \frac{1}{a}\left([\psi' \circ \psi](az)c_k(az) + [c_k \circ \psi](az)e^{2\pi k\omega i}\right)e^{2\pi k\theta i}. \tag{24}$$

On the other hand, when (9) is evaluated at $c_0(z)$ one has

$$[D\mathcal{T}(\Psi)]\left(c_0(z)\right) = D\mathcal{R}_\delta(\psi)c_0(z),$$

as should be expected since $c_0 = p_0(f)$.

Given U an open subset of $\mathbb{C}$ we will denote by $\mathcal{RH}(U)$ the set of real holomorphic functions in U and continuous in its closure.

Consider the operators,

$$\begin{array}{rccc} L_1: & \mathcal{RH}(\mathbb{W}) & \to & \mathcal{RH}(\mathbb{W}) \\ & g(z) & \mapsto & \dfrac{1}{a}\psi' \circ \psi(az) g(az), \end{array}$$

and

$$\begin{array}{rccc} L_2: & \mathcal{RH}(\mathbb{W}) & \to & \mathcal{RH}(\mathbb{W}) \\ & g(z) & \mapsto & \dfrac{1}{a} g \circ \psi(az), \end{array}$$

with $\psi = p_0(\Psi)$ and $a = \psi(1)$.

Given a function $f \in \mathcal{B}$, we can consider its Fourier expansion (22) and apply (24), hence

$$[D\mathcal{T}_\omega(\Psi) f](\theta, z) = D\mathcal{R}_\delta[c_0](z) + \sum_{k \in \mathbb{Z}\setminus\{0\}} \Big([L_1(c_k)](z) + [L_2(c_k)](z) e^{2\pi k \omega i} \Big) e^{2\pi k \theta i}. \tag{25}$$

Looking at this formula it can be said that $D\mathcal{T}_\omega$ "diagonalizes" with respect to the complex Fourier base.

We define

$$U_k := \big\{ f \in B |\ f(\theta, x) = u(x)\cos(2\pi k\theta), \text{ for some } u \in \mathcal{RH}(\mathbb{W}) \big\},$$

and

$$V_k := \big\{ f \in B |\ f(\theta, x) = v(x)\sin(2\pi k\theta), \text{ for some } v \in \mathcal{RH}(\mathbb{W}) \big\}.$$

On the other hand, given $\omega \in \mathbb{T}$, consider the following operator

$$\begin{array}{rccc} \mathcal{L}_\omega: & \mathcal{RH}(\mathbb{W}) \oplus \mathcal{RH}(\mathbb{W}) & \to & \mathcal{RH}(\mathbb{W}) \oplus \mathcal{RH}(\mathbb{W}) \\ & \begin{pmatrix} u \\ v \end{pmatrix} & \mapsto & \begin{pmatrix} L_1(u) \\ L_1(v) \end{pmatrix} + \begin{pmatrix} \cos(2\pi\omega) & -\sin(2\pi\omega) \\ \sin(2\pi\omega) & \cos(2\pi\omega) \end{pmatrix} \begin{pmatrix} L_2(u) \\ L_2(v) \end{pmatrix}. \end{array} \tag{26}$$

Then we have the following result.

Proposition 2.16. *The spaces $U_k \oplus V_k$ are invariant by $D\mathcal{T}(\Psi)$ for any $k > 0$. Moreover $D\mathcal{T}_\omega(\Psi)$ restricted to $U_k \oplus V_k$ is conjugated to $\mathcal{L}_{k\omega}$.*

Due to this proposition we have that the understanding of the derivative of the renormalization operator in $\mathcal{B}$ is equivalent to the study of the operator $\mathcal{L}_\omega$ for any $\omega \in \mathbb{T}$. Therefore we focus now on the study of $\mathcal{L}_\omega$.

Given a value $\gamma \in \mathbb{T}$, consider the rotation R_γ defined as

$$\begin{array}{rccc} R_\gamma: & \mathcal{RH}(\mathbb{W}) \oplus \mathcal{RH}(\mathbb{W}) & \to & \mathcal{RH}(\mathbb{W}) \oplus \mathcal{RH}(\mathbb{W}) \\ & \begin{pmatrix} u \\ v \end{pmatrix} & \mapsto & \begin{pmatrix} \cos(2\pi\gamma) & -\sin(2\pi\gamma) \\ \sin(2\pi\gamma) & \cos(2\pi\gamma) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \end{array} \tag{27}$$

Then we have the following result.

Proposition 2.17. *For any $\omega, \gamma \in \mathbb{T}$ we have that $\mathcal{L}_\omega$ and R_γ commute.*

This proposition has the following spectral consequences on $\mathcal{L}_\omega$.

Corollary 2.18. *For any eigenvector (u, v) of $\mathcal{L}_\omega$ we have that $R_\gamma(u, v)$ is also an eigenvector of the same eigenvalue for any $\gamma \in \mathbb{T}$.*

Corollary 2.19. *All the eigenvalues of $\mathcal{L}_\omega$ (different from zero) are either real with geometric multiplicity even, or a pair of complex conjugate eigenvalues.*

On the other hand we have the following result on the dependence of the operator with respect to ω

Proposition 2.20. *The operator $\mathcal{L}_\omega$ depends analytically on ω.*

This result allows us to apply theorems III-6.17 and VII-1.7 of [14]. These results imply that, as long as the eigenvalues of $\mathcal{L}_\omega$ do not cross each other, then the eigenvalues and their associated eigenspaces depend analytically on the parameter ω.

We want to prove the compactness of $\mathcal{L}_\omega$ as an operator. For technical reasons this can not be proved with $\mathcal{L}_\omega$ as an operator on $\mathcal{RH}(\mathbb{W}) \oplus \mathcal{RH}(\mathbb{W})$, but it can be proved on a closed subspace of $\mathcal{RH}(\mathbb{W}) \oplus \mathcal{RH}(\mathbb{W})$.

Proposition 2.21. *Consider $K \subset \mathbb{W}$ a compact set in the complex plane, such that $\psi(a\mathbb{W}) \subset K$ and $a\mathbb{W} \subset K$ where $\psi = p_0(\Psi)$. Let us denote by $B = \mathcal{RH}(\mathbb{W}) \cap C^0(K, \mathbb{C})$, which is a Banach subspace of $\mathcal{RH}(\mathbb{W})$.*

Then the operator $\mathcal{L}_\omega$ restricted to the subspace $B \oplus B \subset \mathcal{RH}(\mathbb{W}) \oplus \mathcal{RH}(\mathbb{W})$ is well defined (i.e. $\mathcal{L}_\omega : B \oplus B \to B \oplus B$) and it is compact.

Recall that the compacity of an operator implies that its spectrum is either finite or countable with 0 on its closure (see for instance theorem III-6.26 of [14]).

To finish this section we have included in figure 3 a numerical approximation of the spectrum of the operator $\mathcal{L}_\omega$ depending on ω. In the figure it can be observed that the different properties (and their spectral consequences) on the operator stated above are satisfied. The details on the numerical computations involved to approximate the spectrum are described in [23]. Several numerical tests on the reliability of the results are also included there.

Proofs

Proof of proposition 2.16. Let f be a function in $U_k \oplus V_k$, then we have $f(\theta, x) = u(z)\cos(2\pi k\theta) + v(z)\sin(2\pi k\theta)$. Consider the function $c(z) = \frac{u(z)+iv(z)}{2}$. Using formula (24) on the function $u(z) = c(z) + \bar{c}(z)$ and doing some algebra one can see that

$$\begin{aligned}
[D\mathcal{T}_\omega(\Psi)](u(z)\cos(2\pi k\theta)) \quad = \quad & [L_1(u)](z)\cos(2\pi k\theta) + [L_2(u)](z)\cos(2\pi k\omega)\cos(2\pi k\theta) \\
& -[L_2(v)](z)\sin(2\pi k\omega)\sin(2\pi k\theta),
\end{aligned}$$

and, doing a similar calculation for $v(z) = i(\overline{c(z)} - c(z))$

$$\begin{aligned}
[D\mathcal{T}_\omega(\Psi)](v(z)\sin(2\pi k\theta)) \quad = \quad & [L_1(v)](z)\sin(2\pi k\theta) + [L_2(u)](z)\sin(2\pi k\omega)\cos(2\pi k\theta) \\
& +[L_2(v)](z)\cos(2\pi k\omega)\sin(2\pi k\theta).
\end{aligned}$$

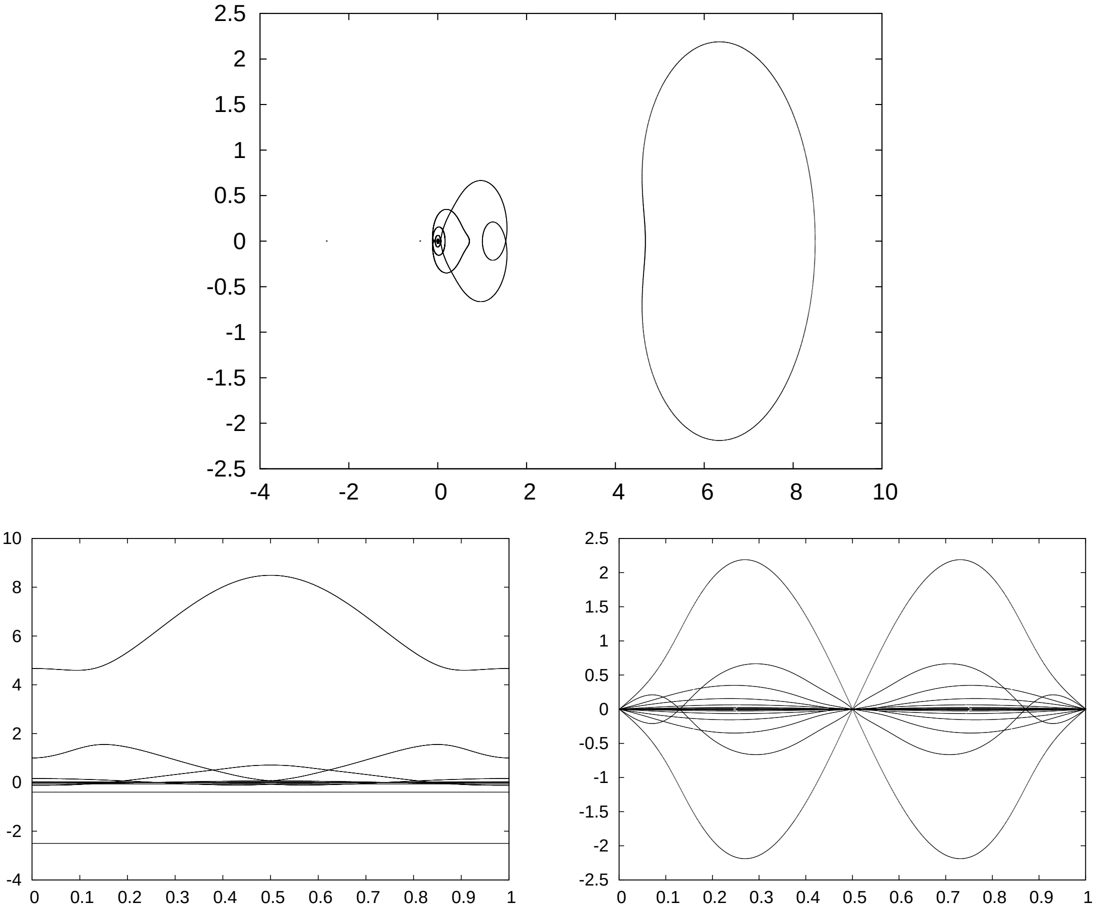

Figure 3: Numerical approximation of the spectrum of $\mathcal{L}_\omega$ for $\omega \in \mathbb{T}$. On the top we have the projection in the complex plane of the spectrum when ω varies in $\mathbb{T}$. In the bottom we have the evolution of the real (left) and the imaginary (right) part of the spectrum with respect to ω.

Now notice that there is a natural isomorphism between U_k and $\mathcal{RH}(\mathbb{W})$ given by the function $p_c : \mathcal{RH}(\mathbb{W}) \to U_k$ defined as $p_c(f)(x) := \int_0^1 f(\theta, x) \cos(2\pi k\theta) d\theta$ and the function $i_c : U_k \to \mathcal{RH}(\mathbb{W})$ defined as $i_c(f)(\theta, x) = f(x)\cos(2\pi k\theta)$. The same argument can be applied to V_k considering the functions $p_s : V_k \to \mathcal{RH}(\mathbb{W})$ and $i_s : \mathcal{RH}(\mathbb{W}) \to V_k$ defined as before but with $\sin(2\pi k\theta)$ instead of $\cos(2\pi k\theta)$. Then these functions can be used to define the isomorphic conjugacy between $D\mathcal{T}_\omega(\Psi)$ and $\mathcal{L}_{k\omega}$. $\square$

Proof of proposition 2.17. It follows from L_1 and L_2 being linear and the fact that any pair of rotations commute. $\square$

Proof of corollary 2.18. Suppose that (u, v) is an eigenvector of eigenvalue λ, we have $\lambda(u, v) = \mathcal{L}_\omega(u, v)$. Composing in both parts by R_γ and using the last proposition the result follows. $\square$

Proof of corollary 2.19. In the case of a real eigenvalue, suppose it has geometric multiplicity odd, then its eigenspace is generated by n vectors $y_1, y_2, \ldots, y_n$, with n odd. We can consider $R_\gamma y_i$ for any i, which will also be in the eigenspace of the eigenvalue. Since the vector $R_\gamma y_i$ is linearly independent with y_i but it is in the eigenspace, we have that it is generated by the

other eigenvectors. Then one of the original vectors can be replaced by $R_\gamma y_i$. Rearranging the vectors if necessary we can suppose that $y_2 = R_\gamma y_1$. Doing this process repeatedly we will end up with an even number of vectors.

In the case of a complex eigenvalue, using that the operator $\mathcal{L}_\omega$ is real, if it has a complex eigenvalue λ with eigenvector $v_r + iv_i$, then $\bar{\lambda}$ will also be an eigenvalue with eigenvector $v_r - iv_i$. Given a complex pair of conjugate eigenvalues, the restriction of the operator to the corresponding eigenspace can be written as a uniform scaling composed with a rotation. It can happen that the space generated by these two vectors is invariant by the rotation R_γ introduced before. Then the multiplicity of the eigenvalue can be odd. Actually, if the pair of complex eigenvalues are simple, then there exists a $\gamma_0 \in \mathbb{T}$ such that the rotation associated to the pair of eigenvectors is R_{γ_0}. $\square$

Proof of proposition 2.20. It follows from the fact that $\mathcal{L}_\omega$ is the sum of two bounded linear operators (which do not depend on ω) times an entire function on ω. $\square$

Proof of proposition 2.21. Note that it is enough to prove that the operators L_1 and L_2 are well defined and compact.

Given a map in $g \in B = \mathcal{RH}(\mathbb{W}) \cap C^0(K_1, \mathbb{C})$, consider $[L_1(g)](z) = \frac{1}{a}\psi' \circ \psi(az) g(az)$. Since $\mathrm{Cl}\,(a\mathbb{W}) \subset K$ then the map $g(a\cdot)$ is in $\mathcal{RH}(\mathbb{W}) \cap C^0(K, \mathbb{C})$. Therefore, the map $L_1(g)$ belongs to B which means that $L_1 : B \to B$ is well defined. On the other hand we have that for any $g \in B$, $L_2(g)$ is defined as $[L_2(g)](z) = \frac{1}{a} g \circ \psi(az)$. As the set K has been considered such that $\psi(a\mathbb{W}) \subset K$, L_2 is also well defined.

Consider U the unit ball of B. Since B is a Banach space, to prove that L_i is compact it is enough to prove that $L_i(U)$ is relatively compact (for $i = 1, 2$). This follows easily using proposition 9.13.1 of [5]. For each compact set in K' in $\mathbb{W}$ we have that the maps $L_i(U)$ are bounded, then it follows that $L_i(U)$ is relatively compact in $C^0(K', \mathbb{C})$. Concretely, we can take $K' = K$ and we have that $L_i(U) \subset \mathcal{RH}(\mathbb{W})$ is relatively compact in $C^0(K, \mathbb{C})$, therefore it is relatively compact in B. $\square$

3 Reducibility loss and quasi-periodic renormalization

In this section we use the renormalization operator to study the reducibility loss bifurcations of a two parametric family of q.p. forced map. Concretely, the main result of this section is a proof of the existence of reducibility loss bifurcations for a two-parametric family of q.p. forced map satisfying suitable conditions, see theorem 3.8. In particular this theorem applies to the case of the Forced Logistic Map considered in [12], but this will be discussed in [23]. The proof is not complete, in the sense that we need to assume the injectiveness of the renormalization operator $\mathcal{T}_\omega$ (see conjecture **A**).

In section 3.1 we consider certain sets $\Upsilon_n^+(\omega)$ and $\Upsilon_n^-(\omega)$ of codimension one in the space $\mathcal{B}$, which correspond to the reducibility loss of the attracting 2^n-periodic orbit. We show that the intersection of these sets with the subset of uncoupled maps corresponds to the sets Σ_n of maps (in the one dimensional case) such that its attracting 2^n periodic orbit is super-attracting. The main result of this section relates the sets $\Upsilon_n^+(\omega)$ (respectively $\Upsilon_n^-(\omega)$) for different values of n through the renormalization operator $\mathcal{T}_\omega$.

In section 3.2 we consider a generic two parametric family of maps satisfying certain hypotheses. We intersect the family with the previous $\Upsilon_n^+(\omega)$ and $\Upsilon_n^-(\omega)$. Differently to the one dimensional analogous, the intersection gives a one dimensional curve in the family, which corresponds to a reducibility loss bifurcation. Using the (quasi-periodic) renormalization operator and the results in section 3.1 we prove the existence of reducibility loss bifurcation curves on the parameter space of the family considered. In theorem 3.8 we show that, given a two parametric family which uncouples, there exist two reducibility loss bifurcation curves around the points such that the uncoupled map has a super-attracting periodic orbit. This fact was observed numerically in [12].

We also discus the weakening of the hypothesis of theorem 3.8 and we give explicit expressions of the slopes of the reducibility loss bifurcations in terms of the family of maps and its iterates by the (quasi-periodic) renormalization operator.

As in previous sections the proofs have been moved to the end of each subsection.

3.1 Boundaries of reducibility

In this section we work in the analytic framework, concretely we consider maps belonging to $\mathcal{B}$ the space of the q.p forced one dimensional unimodal maps (as defined in subsection 2.3). Again, let us consider the splitting $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_0^c$ given by the projection $p_0[f](x) := \int_0^1 f(\theta, x)d\theta$, in other words the spaces given by $\mathcal{B}_0 = p_0(\mathcal{B})$ and $\mathcal{B}_0^c = (\mathrm{Id} - p_0)(\mathcal{B})$, where Id is the identity map. The renormalization operator for q.p. forced maps is denoted by $\mathcal{T}_\omega$, the renormalization operator for one dimensional maps by $\mathcal{R}$ and the fixed point by Φ independently of the operator (recall that the fixed points of $\mathcal{R}$ extend automatically to fixed points of $\mathcal{T}_\omega$). Given a map F like (3) with $f \in \mathcal{B}$ and $\omega \in \mathbb{T}$ we denote by $f^n : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ the x-projection of $F^n(x, \theta)$. Equivalently, f^n can be defined through the recurrence

$$f^n(\theta, x) = f(\theta + (n-1)\omega, f^{n-1}(\theta, x)). \tag{28}$$

In this subsection, differently to the previous one, whenever ω is used, it is assumed to be Diophantine. Let us denote by Ω the set of Diophantine numbers, this is $\Omega = \Omega_{\gamma,\tau}$ the set of $\omega \in \mathbb{T}$ such that there exists $\gamma > 0$ and $\tau \geq 1$ such that

$$|qw - p| \geq \frac{\gamma}{|q|^\tau}, \qquad \text{for all } (p, q) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}).$$

Additionally, we assume that the following conjecture is true.

Conjecture A. *The operator $\mathcal{T}_\omega$ (for any $\omega \in \Omega$) is an injective function when restricted to the domain $\mathcal{B} \cap \mathcal{D}(\mathcal{T})$. Moreover, there exist U an open set of $\mathcal{D}(\mathcal{T})$ containing $W^u(\Phi, \mathcal{R}) \cup W^s(\Phi, \mathcal{R})$*[1] *where the operator $\mathcal{T}_\omega$ is differentiable.*

The first part of the conjecture is proved for the one dimensional case in [4]. The proof consists of, given two maps with the same image by the operator, first to show that their renormalization interval is the same and then to expand the image of the maps around their fixed point and then deduce that the original maps are the same maps. With our approach to the quasi-periodic case, as we do not have an equivalent concept to renormalization interval, the same

[1]Here $W^s(\Phi, \mathcal{R})$ and $W^u(\Phi, \mathcal{R})$ are considered as the inclusion in $\mathcal{B}$ of the stable and the unstable manifolds of the fixed point Φ (given by **H0**) by the map $\mathcal{R}$ in the topology of $\mathcal{B}_0$ (the inclusion of one parametric maps in $\mathcal{B}$).

argument is no longer applicable. Despite the analogy with the one dimensional case, there is, a priori, no evidence about the conjecture. A posteriori, we have that the results obtained assuming that the conjecture is true are coherent with the dynamics of the Forced Logistic Map. In [23] we compute numerically the slopes of the reducibility loss bifurcations of the FLM by two independent methods. The first method is computing the slope using the dynamical characterization of the bifurcations. The second one is using the formulas given in corollary 3.13. Both coincide up to a reasonable accuracy.

The second part of the conjecture is only introduced to simplify the forthcoming discussion, but it can be avoided if necessary. See remark 3.9 for details.

Whenever the conjecture **A** is needed for a result it is explicitly stated in the hypotheses.

Let $K_0 > 0$ be a fixed constant value. Then we can consider the sets

$$\Upsilon_n^+(\omega) = \left\{ f \in \mathcal{B} \;\middle|\; \begin{array}{c} \text{There exists } x_0 \in \mathcal{RH}(\mathbb{B}_\rho, \mathbb{W}) \text{ with } x_0(\theta + 2^n\omega) = f^{2^n}(\theta, x_0(\theta)) \text{ s. t.} \\ \displaystyle\int_0^1 \ln |D_x f^{2^n}(\theta, x_0(\theta))| d\theta < -K_0 \text{ and } \min_{\theta\in\mathbb{T}} D_x \left(f^{2^n}\right)(\theta, x_0(\theta)) = 0. \end{array} \right\},$$

and

$$\Upsilon_n^-(\omega) = \left\{ f \in \mathcal{B} \;\middle|\; \begin{array}{c} \text{there exists } x_0 \in \mathcal{RH}(\mathbb{B}_\rho, \mathbb{W}) \text{ with } x_0(\theta + 2^n\omega) = f^{2^n}(\theta, x_0(\theta)) \text{ s. t.} \\ \displaystyle\int_0^1 \ln |D_x f^{2^n}(\theta, x_0(\theta))| d\theta < -K_0 \text{ and } \max_{\theta\in\mathbb{T}} D_x \left(f^{2^n}\right)(\theta, x_0(\theta)) = 0. \end{array} \right\}.$$

Note that (due to the two first conditions) these sets are contained in the set of all the maps in $\mathcal{B}$ which have a 2^n-periodic attracting curve. We require the integral being less than $-K_0$ instead of being less than 0 for technical reasons. The third condition is imposed with the aim that these sets correspond to the bifurcation manifold associated to the reducibility loss. We have the following properties which give a good characterization of these sets.

Proposition 3.1. *Let Σ_n be the inclusion in $\mathcal{B}$ of the set of one dimensional unimodal maps with a super-attracting 2^n periodic orbit. We have that*

$$\Upsilon_n^+(\omega) \cap \mathcal{B}_0 = \Upsilon_n^-(\omega) \cap \mathcal{B}_0 = \Sigma_n,$$

for any $\omega \in \Omega$.

Proposition 3.2. *Let $f \in \Upsilon_n^+(\omega)$ (respectively $f \in \Upsilon_n^-(\omega)$) and let x be its 2^n-periodic curve. If $D_x \left(f^{2^n}\right)(\theta, x(\theta))$ has a unique non-degenerate absolute minimum[2] (respectively maximum), then $\Upsilon_n^+(\omega)$ (respectively $f \in \Upsilon_n^-(\omega)$) is a codimension one manifold in a neighborhood of f.*

Proposition 3.3. *Let $\{f_\mu\}_{\mu\in A}$ be a one parametric family of maps such that:*

1. *There exist a parameter value μ_0 for which the family crosses $\Upsilon_n^+(\omega)$ (respectively $\Upsilon_n^-(\omega)$) transversely at $\mu = \mu_0$.*

2. *Consider x_{μ_0} the 2^n periodic invariant curve of f_{μ_0} given by the definition of $\Upsilon_n^+(\omega)$ (respectively $\Upsilon_n^-(\omega)$) such that $D_x \left(f_{\mu_0}^{2^n}\right)(\theta, x_{\mu_0}(\theta))$ has a unique non-degenerate minimum (respectively maximum).*

[2] It can have several local minima but the absolute minimum has to be unique and not degenerate.

Then we have that the invariant periodic curve x_{μ_0} extends to a periodic invariant curve[3] x_μ *of f_μ for any μ in an open neighborhood of μ_0. Additionally, this invariant curve undergoes a reducibility loss bifurcation at $\mu = \mu_0$.*

Let us introduce some notation to state the next result. Consider the map,

$$\begin{array}{rccc} T: & \mathbb{T} \times \mathcal{D}(\mathcal{T}) & \to & \mathbb{T} \times \mathcal{B} \\ & (\omega, f) & \mapsto & (2\omega, \mathcal{T}_\omega(f)), \end{array} \tag{29}$$

where $\mathcal{T}_\omega$ is the renormalization operator for q.p. forced maps, as in section 2.2, and the set $\mathcal{D}(\mathcal{T}) \subset \mathcal{B}$ is its domain of definition. Recall that to have $\mathcal{T}_\omega$ well defined it is not necessary $\omega \in \Omega$ (i.e. ω Diophantine). Additionally, if $\omega \in \Omega$ then we have that $2^k\omega \in \Omega$ for any $k \in \mathbb{Z}$, therefore $T(\Omega \times \mathcal{D}(\mathcal{T})) \subset \Omega \times \mathcal{B}$. On the other hand we have that the sets $\Upsilon_n^+(\omega)$ and $\Upsilon_n^-(\omega)$ are only defined for $\omega \in \Omega$.

Definition 3.4. We will say that a pair $(\omega, f) \in \mathbb{T} \times \mathcal{B}$ is **n-times renormalizable** if $T^k(\omega, f) \in \mathbb{T} \times \mathcal{D}(\mathcal{T})$ for $k = 0, \ldots, n-1$.

Consider a pair $(\omega, f_0) \in \Omega \times \mathcal{B}$ with a 2-periodic invariant attracting curve x_0 with rotation number ω. Assume that the Lyapunov exponent of the curve is less that $-K_0$, with $K_0 > 0$ a fixed value, in other words,

$$\int_0^1 \ln |D_x f_0^2(\theta, x_0(\theta))| d\theta < -K_0.$$

In forthcoming lemma 3.6 we will prove that the persistence of the invariant curve extends to a neighborhood of f_0 (if the width ρ of the band $\mathbb{B}_\rho$ around the torus $\mathbb{T}$ is small enough with respect to K_0). Let $V \subset \mathcal{B}$ be this neighborhood, and let $x : V \to \mathcal{RH}(\mathbb{B}_\rho, \mathbb{C})$, be the period 2 invariant curve associated to f. Here $\mathcal{RH}(\mathbb{B}_\rho, \mathbb{C})$ denotes the space of functions $f : \mathbb{B}_\rho \to \mathbb{C}$ which are real analytic in $\mathbb{B}_\rho$ and continuous in its closure. Then we can define the map G_1 as

$$\begin{array}{rccc} G_1: & \Omega \times V & \to & \mathcal{RH}(\mathbb{B}_\rho, \mathbb{C}) \\ & (\omega, g) & \mapsto & D_x g\big(\theta + \omega, g(\theta, [x(\omega, g)](\theta))\big) D_x g\big(\theta, [x(\omega, g)](\theta)\big). \end{array} \tag{30}$$

On the other hand, consider the minimum and the maximum as operator between spaces of functions:

$$\begin{array}{rccc} m: & \mathcal{RH}(\mathbb{B}_\rho, \mathbb{C}) & \to & \mathbb{R} \\ & g & \mapsto & \min_{\theta \in \mathbb{T}} g(\theta). \end{array} \tag{31}$$

and

$$\begin{array}{rccc} M: & \mathcal{RH}(\mathbb{B}_\rho, \mathbb{C}) & \to & \mathbb{R} \\ & g & \mapsto & \max_{\theta \in \mathbb{T}} g(\theta). \end{array} \tag{32}$$

We have the following theorem, which relates the manifolds $\Upsilon_n^+(\omega)$ and $\Upsilon_n^-(\omega)$ for different n through the renormalization operator.

Theorem 3.5. *Let $\omega \in \Omega$ and $f \in \Upsilon_n^+(\omega)$, respectively $f \in \Upsilon_n^-(\omega)$, be a function such that the pair (ω, f) is $n-1$ times renormalizable. Additionally assume that conjecture* **A** *is true. Then there exist U a neighborhood (in $\mathcal{B}$) of f such that*

$$U \cap \Upsilon_n^+(\omega) = \{f \in U |\, G_1^+(T^{n-1}(\omega, f)) = 0\},$$

[3] To extend the periodic invariant curve we need to reduce ρ (the width of the band of analyticity with respect to θ) to be small enough in terms of K_0, but this reduction of ρ is done only once.

respectively

$$U \cap \Upsilon_n^-(\omega) = \{f \in U | \, G_1^-(T^{n-1}(\omega, f)) = 0\},$$

where

$$G_1^+(\omega, g) := m \circ G_1(\omega, g),$$

and respectively

$$G_1^-(\omega, g) := M \circ G_1(\omega, g),$$

with G_1, m and M defined in equations (30), (31) and (32).

Proofs

Proof of proposition 3.1. We will do the proof only for the case $\Upsilon_n^+(\omega)$. The case $\Upsilon_n^-(\omega)$ is completely analogous.

If we have a map $f_0 \in \Sigma_n \subset \mathcal{B}_0$, it has a super-attracting periodic orbit x_0, then we have that its Lyapunov exponent is $-\infty$, since $D_x f^{2^n}(\theta, x_0) \equiv 0$. Therefore $\Sigma_n \subset \Upsilon_n^+(\omega) \cap \mathcal{B}_0$.

Consider $f \in \Upsilon_n^+ \cap \mathcal{B}_0$. Since f is in $\mathcal{B}_0$ it does not depend on θ. Consider also $h = f^{2^n}$, which neither depends on θ. On the other hand using that f is in $\Upsilon_n^+(\omega)$ we have that there exists a function $x \in \mathcal{RH}(\mathbb{B}_\rho, \mathbb{W})$ satisfying the following invariance equation

$$x(\theta + 2^n\omega) = h(x(\theta)).$$

Differentiating the invariance equation we have

$$x'(\theta + 2^n\omega) = h'(x(\theta))x'(\theta).$$

From the fact that $f \in \Upsilon_n^+(\omega)$ it follows that there exist a θ_0 such that $h'(x(\theta_0)) = 0$. Using the last equation we have that x' is zero in a dense subset of $\mathbb{T}$ and using the continuity of x' we have that $x' \equiv 0$, therefore x is constant. Finally note that as $x = h(x) = f^{2n}(x)$ and $D_x f^{2^n}(x) = 0$, we have that f belongs to Σ_n. □

Proof of proposition 3.2. As before we prove only the case of Υ_n^+. We start with a preliminary lemma.

Lemma 3.6. *Consider $g_0 \in \mathcal{B} = \mathcal{B}(\mathbb{B}_\rho, \mathbb{W})$ and x_0 an invariant curve of g_0. Assume that*

$$\int_0^1 \ln |D_x g_0(\theta, x_0(\theta))| d\theta < -K < 0. \tag{33}$$

Then, for a sufficiently small value of ρ, there exist a neighborhood U of g_0 and a smooth function $x : U \to \mathcal{B}$ such that $x(g)$ is an invariant curve of g for any $g \in U$ and $x(g_0) = x_0$.

Proof. This lemma corresponds to the analytic version of the continuation problem of an invariant curve. In [13] it is studied the C^r version of this problem. The authors prove that the curve can be continued if 1 does not belong to the spectrum of the transfer operator $\mathcal{L}$ associated to the problem (see section 3.3 of [13]). Then it is shown that the spectrum of $\mathcal{L}$ is contained in the disk of radius $b = \exp\left(\int_0^1 \ln |D_x g_0(\theta, x_0(\theta))| d\theta\right)$.

Let $\mathbb{B}_\rho$ be a band of width ρ around the real torus $\mathbb{T}$, and $\mathcal{H}(\mathbb{B}_\rho, \mathbb{C})$ denote the space of holomorphic maps from $\mathbb{B}_\rho$ to $\mathbb{C}$ and continuous on the closure of $\mathbb{B}$. Note that transfer operator $\mathcal{L}$ can

be considered both, in $C^r(\mathbb{T},\mathbb{R})$ endowed with the standard C^r-norm, or in $\mathcal{H}(\mathbb{B}_\rho,\mathbb{C})$ endowed with the supremum norm. To distinguish the spectrum of the transfer operator with respect to the norm considered we will denote each of the respective cases by $\mathrm{Spec}(\mathcal{L},C^r)$ or $\mathrm{Spec}(\mathcal{L},\mathcal{H})$. Using theorem 9.2 of [9], we have that

$$\partial\mathrm{Spec}(\mathcal{L},\mathcal{H}) \subset \mathrm{Spec}(\mathcal{L},C^r) + O(\rho),$$

for $\rho > 0$ small enough.

The notation $A \subset B + O(\rho)$ means that there exists a constant $C > 0$ such that for any $a \in A$ there exists $b \in B$ with $d(a,b) \leq C\rho$.

Using equation (33) we have that $b < 1$. Then there exists a sufficiently small ρ such that $b + \rho < 1$. Then using that $\partial\mathrm{Spec}(\mathcal{L},\mathcal{H})$ is contained in the disc of radius $b + \rho < 1$ one can extend the persistence of invariant attracting curves to the analytic case. $\square$

Consider an arbitrary function $f_0 \in \Upsilon_n^+(\omega)$. We have that there exists $x_0 : \mathbb{B}_\rho \to \mathbb{W}$ which is a 2^n-invariant attracting curve of f_0. We can consider the auxiliary function

$$\begin{array}{rccc} F: & \mathcal{B} \times \mathcal{RH}(\mathbb{B}_\rho,\mathbb{W}) & \to & \mathcal{RH}(\mathbb{B}_\rho,\mathbb{W}) \\ & (g,x) & \mapsto & [F(g,x)](\theta) := x(\theta + 2^n\omega) - g^{2^n}(\theta, x(\theta)). \end{array}$$

We have that the Lyapunov exponent of the curve is less than $-K_0$. Using the lemma 3.6 we have that, if the width ρ of the band $\mathbb{B}_\rho$ around the torus $\mathbb{T}$ is small enough with respect to K_0, then there exists a neighborhood U_n of f_0 in $\mathcal{B}$ and a function

$$\begin{array}{rccc} x: & U_n & \to & \mathcal{RH}(\mathbb{B}_\rho,\mathbb{W}) \\ & g & \mapsto & x(g), \end{array}$$

with $x(f_0) = x_0$ and such that $x(f)$ is a 2^n periodic curve of f for any $f \in U_n$. Moreover, due to the continuity of the Lyapunov exponent, we can suppose that the Lyapunov exponent of $x(g)$ is negative for any $g \in U_n$, replacing U_n by a smaller neighborhood if necessary.

Now we consider the auxiliary function

$$\begin{array}{rccc} \tilde{G}_n: & U_n & \to & \mathcal{RH}(\mathbb{B}_\rho,\mathbb{C}) \\ & f & \mapsto & [\tilde{G}_n(f)](\theta) := D_x\left(f^{2^n}\right)(\theta, [x(f)](\theta)), \end{array} \tag{34}$$

and consider also the minimum operator defined as in (31).

By hypothesis we have that $\left[D_x\left(f_0^{2^n}\right)\right](\theta,[x(f_0)](\theta))$ has a unique minimum, therefore we can apply proposition A.1 (in appendix A). The uniqueness of the minimum extends to U_n, replacing it again by a smaller neighborhood if necessary. Therefore we have that the map $G_n : U_n \to \mathbb{R}$ defined as $m \circ \tilde{G}_n$ is differentiable. Let us remark that G_n depends indeed on ω, but the differentiability is only needed with respect to g. Then we have that

$$\Upsilon_n^+(\omega) \cap U_0 = \{f \in U |\, G_n(\omega, f) = 0\}, \tag{35}$$

which completes the proof. $\square$

Proof of proposition 3.3. As before we only consider the case involving $\Upsilon_n^+(\omega)$. Consider f_{μ_0} the intersection of the family with the set $\Upsilon_n^+(\omega)$. Using the second hypothesis of the proposition

we are under the same hypothesis of proposition 3.2. Following the proof of this proposition we have that there exists U_0 an open neighborhood of f_{μ_0} such that $\Upsilon_n^+(\omega) \cap U_0$ is given by equation (35) with $G_n = m \circ \tilde{G}_n$ and the maps m and $\tilde{G}_n$ are given by equations (31) and (34). Moreover we also have that G_n is differentiable in U_0.

Using that the family f_μ crosses transversely the manifold $\Upsilon_n^+(\omega)$ we have that $\partial_\mu G_n(f_\mu)_{|\mu=\mu_0} \neq 0$. Actually we can assume that $\partial_\mu G_n(f_\mu)_{|\mu=\mu_0} < 0$, otherwise we can replace μ by $\tilde{\mu} = 2\mu_0 - \mu$ and consider the family $f_{\tilde{\mu}}$ instead of f_μ. Recall also that

$$G_n(f_\mu) = \min_{\theta \in \mathbb{T}} D_x \left(f_\mu^{2^n}\right)(\theta, x_\mu(\theta)).$$

Then for $\mu < \mu_0$ we have $G_n(f_\mu) > 0$ and therefore we have that $D_x \left(f_{\mu_0}^{2^n}\right)(\theta, x_{\mu_0}(\theta)) > 0$ for any $\theta \in \mathbb{T}$. Using corollary 1 of [13] we have that x_μ is reducible. Moreover, due to second hypothesis of the proposition we have that $D_x \left(f_{\mu_0}^{2^n}\right)(\theta, x_{\mu_0}(\theta))$ has a double zero θ_0. Finally using the transversality hypothesis we have that $\partial_\mu \left(D_x \left(f_{\mu_0}^{2^n}\right)(\theta_0, x_{\mu_0}(\theta_0))\right) \neq 0$. This proves that x_μ undergoes a reducibility loss bifurcation. $\square$

Proof of theorem 3.5. Once again, we only consider the case involving $\Upsilon_n^+(\omega)$, since the other case is completely analogous. Let us introduce the following lemma for the proof of theorem.

Lemma 3.7. *Let* $\mathrm{Im}(\mathcal{T}_\omega)$ *be the image of the operator* $\mathcal{T}_\omega$*. Assume that we have* $\omega \in \Omega$*, then we have that*

$$\mathcal{T}_\omega(\Upsilon_n^+(\omega) \cap \mathcal{D}(\mathcal{T})) = \Upsilon_{n-1}^+(2\omega) \cap \mathrm{Im}(\mathcal{T}_\omega). \tag{36}$$

Proof. Given $f \in \mathcal{T}_\omega(\Upsilon_n^+(\omega) \cap \mathcal{D}(\mathcal{T}))$, we have that there exists a function $g \in \Upsilon_n^+(\omega)$ such that

$$f(\theta, x) = \frac{1}{a} g(\theta + \omega, g(\theta, ax)), \tag{37}$$

with $a = \int_0^1 g(\theta, 1) d\theta$. Note that the rotation number when we compose g with itself is ω, while the rotation number when we compose f with itself is 2ω; this is not obvious from the notation but it is important to have it in mind for this proof.

To prove the first inclusion it is enough to check that f is in $\Upsilon_{n-1}^+(2\omega)$ since f is trivially in $\mathrm{Im}(\mathcal{T}_\omega)$.

Using that g is in $\Upsilon_n^+(\omega)$ we have that there exists $x_0 : \mathbb{B}_\rho \to \mathbb{W}$ with

$$x_0(\theta + 2^n\omega) = g^{2^n}(\theta, x_0(\theta)),$$

and such that

$$\min_{\theta \in \mathbb{T}} D_x \left(g^{2^n}\right)(\theta, x_0(\theta)) = 0.$$

Using equation (37) it is easy to check that

$$f^{2^{n-1}}(\theta, x) = \frac{1}{a} g^{2^n}(\theta, ax), \tag{38}$$

and

$$D_x \left(f^{2^{n-1}}\right)(\theta, x) = D_x \left(g^{2^n}\right)(\theta, ax), \tag{39}$$

for any $\theta \in \mathbb{B}_\rho$ and $x \in \mathbb{W}$.

Consider x_1 the function defined as $x_1(\theta) = \frac{1}{a}x_0(\theta)$. From the last two equalities it follows that $x_1(\theta + 2^{n-1}(2\omega)) = f^{2^{n-1}}(\theta, x_1(\theta))$, and

$$\min_{\theta\in\mathbb{T}} D_x \left(f^{2^{n-1}}\right)(\theta, x_1(\theta)) = \min_{\theta\in\mathbb{T}} D_x \left(g^{2^n}\right)(\theta, x_0(\theta)) = 0.$$

Therefore f is in $\Upsilon^+_{n-1}(2\omega)$.

Let us see the converse inclusion. Consider $f \in \Upsilon^+_{n-1}(2\omega)\cap\mathrm{Im}(\mathcal{T}_\omega)$. Since f is in $\mathrm{Im}(\mathcal{T}_\omega)$ we have that there exists $g \in \mathcal{D}(\mathcal{T})$ with $f = \mathcal{T}_\omega(g)$. Therefore we only have to prove that g is in $\Upsilon^+_n(\omega)$. Using $f = \mathcal{T}_\omega(g)$ one has that equation (37) is satisfied again and this implies that equations (38) and (39) also hold. From $f \in \Upsilon^+_{n-1}(2\omega)$ we have that there exists a function $x_1 : \mathbb{B}_\rho \to \mathbb{W}$ with $x_1(\theta + 2^{n-1}(2\omega)) = f^{2^{n-1}}(\theta, x_1(\theta))$ and

$$\min_{\theta\in\mathbb{T}} D_x \left(f^{2^{n-1}}\right)(\theta, x_1(\theta)) = 0.$$

Consider now $x_0(\theta) := ax_1(\theta)$, then using equation (38) we have that

$$ax_1(\theta + 2^{n-1}(2\omega)) = g^{2^n}(\theta, ax_1(\theta)),$$

for any $\theta \in \mathbb{T}$. Using (39) we obtain

$$\min_{\theta\in\mathbb{T}} D_x \left(g^{2^n}\right)(\theta, x_1(\theta)) = 0,$$

what completes the proof of the lemma. □

The proof will follow by induction. Note that the case $n = 1$ is satisfied trivially. Then we can assume that the case $n - 1$ is true and check the case n.

We have that (f, ω) is renormalizable, then $f \in D(\mathcal{T}) \cap \Upsilon^+_n(\omega)$. Using conjecture **A** we have that a point belongs to $\Upsilon^+_n(\omega)\cap\mathcal{D}(\mathcal{T})$ if, and only if, $\mathcal{T}_\omega(g)$ belongs to $\mathcal{T}_\omega(\Upsilon^+_n(\omega)\cap\mathcal{D}(\mathcal{T}))$. Using lemma 3.7 we have that $\mathcal{T}_\omega(\Upsilon^+_n(\omega) \cap \mathcal{D}(\mathcal{T})) = \Upsilon^+_{n-1}(2\omega) \cap \mathrm{Im}(\mathcal{T}_\omega)$.

At this point we need to consider the case $n = 2$ independently. In the case $n = 2$ we have that $\mathcal{T}_\omega(\Upsilon^+_2(\omega) \cap \mathcal{D}(\mathcal{T})) = \Upsilon^+_1(2\omega) \cap \mathrm{Im}(\mathcal{T}_\omega)$, then we have that there exists U_1 a neighborhood of $\mathcal{T}_\omega(f)$ such that

$$U_1 \cap \Upsilon^+_1(2\omega) = \{g \in U|\, G_1(2\omega, g) = 0\}.$$

Consider $U_2 = \mathcal{T}_\omega^{-1}(U_1)$, using that $\mathcal{T}_\omega$ is continuous, we have that U_2 is an open neighborhood of f. Then we have

$$U_2 \cap \Upsilon^+_2(\omega) = \mathcal{T}_\omega^{-1}(U_1 \cap \Upsilon^+_1(2\omega)) = \{f \in U_1|\, G_1(2\omega, \mathcal{T}_\omega(f)) = G_1(T(\omega, f)) = 0\},$$

which finishes the proof for the case $n = 2$.

In the case $n > 2$ we have that the pair $(2\omega, \mathcal{T}_\omega(f))$ is $n-2$ times renormalizable We apply now the induction hypothesis, then we have that there exists U_{n-1} a neighborhood of $\mathcal{T}_\omega(f)$ such that

$$U_{n-1} \cap \Upsilon^+_{n-1}(2\omega) = \{g \in U|\, G_1(T^{n-2}(2\omega, g)) = 0\}.$$

Consider $U_n = \mathcal{T}_\omega^{-1}(U_{n-1})$, since $\mathcal{T}_\omega$ is a continuous function, we have that U_n is an open neighborhood of f and then we have

$$U_n \cap \Upsilon^+_n(\omega) = \mathcal{T}_\omega^{-1}(U_{n-1} \cap \Upsilon^+_{n-1}(2\omega)) = \{f \in U_{n-1}|\, G_1(T^{n-2}(2\omega, \mathcal{T}_\omega(f))) = 0\}.$$

Using that $T^{n-2}(2\omega, \mathcal{T}_\omega(f)) = T^{n-1}(\omega, f)$ the proof is finished. □

3.2 Consequences for a two parametric family of maps

Consider a two parametric family of maps like (3). For the rest of this section we assume that ω is a fixed Diophantine number ($\omega \in \Omega$). Then the family of maps is determined by two parametric family of maps $\{c(\alpha, \varepsilon)\}_{(\alpha,\varepsilon)\in A}$ contained in $\mathcal{B}$ (concretely they are unimodal q.p. forced maps), where $A = [a, b] \times [0, c]$ and a, b and c are real numbers (with $a < b$ and $0 < c$).

We assume that the dependency on the parameters is analytic, then the family can be thought as an analytic map $c : A \to \mathcal{B}$. In this subsection we prove (under suitable hypotheses) the existence of reducibility loss bifurcations like the ones observed in the numerical computations of the Forced Logistic Map [12, 23].

We will consider families of maps satisfying the following hypothesis.

H1) The family $\{c(\alpha, \varepsilon)\}_{(\alpha,\varepsilon)\in A}$ uncouples for $\varepsilon = 0$, in the sense that the family $\{c(\alpha, 0)\}_{\alpha\in[a,b]}$ does not depend on θ and it is a one parametric family of unimodal maps has a full cascade of period doubling bifurcations. We assume that the family $\{c(\alpha, 0)\}_{\alpha\in[a,b]}$ crosses transversally the stable manifold of Φ the fixed point of the renormalization operator and each of the manifolds Σ_n for any $n \geq 1$, where Σ_n is the inclusion in $\mathcal{B}$ of the set of one dimensional unimodal maps with a super-attracting 2^n periodic orbit.

In other words, we assume that the family $c(\alpha, \varepsilon)$ can be written as,

$$c(\alpha, \varepsilon) = c_0(\alpha) + \varepsilon c_1(\alpha, \varepsilon),$$

with $\{c_0(\alpha)\}_{\alpha\in[a,b]} \subset \mathcal{B}_0$ having a full cascade of period doubling bifurcations.

Given a family $\{c(\alpha, \varepsilon)\}_{(\alpha,\varepsilon)\in A}$ satisfying hypothesis **H1** let α_n be the parameter value for which the uncoupled family $\{c(\alpha, 0)\}_{\alpha\in[a,b]}$ intersects the manifold Σ_n. Note that the critical point of the map $c(\alpha_n, 0)$ is a 2^n-periodic orbit. The main goal of this subsection is to prove that for every parameter value $(\alpha_n, 0)$ there are two curves in the parameter space, one corresponding to a reducibility loss bifurcation and the other one corresponding to a reducibility recover. These curves are born at the point $(\alpha_n, 0)$ of the parameter space.

Consider a map $f_0 \in \mathcal{B}$ and $\omega \in \Omega$, such that f has a periodic invariant curve x_0 of rotation number ω with a Lyapunov exponent less or equal than certain $-K_0 < 0$. Recalling the arguments in the proof of proposition 3.2 we have that there exist a neighborhood $V \subset \mathcal{B}$ of f_0 and a map $x \in \mathcal{RH}(\mathbb{B}_\rho, \mathbb{W})$ such that $x(f)$ is a periodic invariant curve of f for any $f \in U_0$. Concretely, if we have a map $f_0 \in \mathcal{B}$ with a 2-periodic invariant attracting curve, we can define the map $G_1 : \Omega \times V \to \mathcal{RH}(\mathbb{B}_\rho, \mathbb{C})$ as in (30).

On the other hand, we can consider the counterpart of the map G_1 in the uncoupled case. Given a map $f_0 \in \mathcal{B}_0$, consider $U \subset \mathcal{B}_0$ a neighborhood of f_0 in the $\mathcal{B}_0$ topology. Assume that f_0 has a attracting 2-periodic orbit $x_0 \in I$. We have that x depends analytically on the map, therefore it induces a map $x : U \to \mathbb{W}$. Then if we take U small enough we can suppose that there exists an analytic map $x : U \to \mathbb{W}$ such that $x[f]$ is a periodic orbit of period 2.

Now consider the map

$$\begin{array}{rccc} \widehat{G}_1 : & U \subset \mathcal{B}_0 & \to & \mathbb{C} \\ & f & \mapsto & D_x f\big(f(x[f])\big) D_x f\big(x[f]\big). \end{array} \tag{40}$$

Let us remark that the zeros of this map define locally the manifold Σ_1. On the other hand it corresponds to the map G_1 restricted to the space $\mathcal{B}_0$, despite the fact that $\widehat{G}_1(f)$ has to be seen as an element of $\mathcal{RH}(\mathbb{B}_\rho, \mathbb{W})$.

At this point we need to introduce an additional hypothesis on the family $\{c(\alpha,\varepsilon)\}_{(\alpha,\varepsilon)\in A}$. Consider $w_k = 2^k w_0$ for any $k \geq 0$ and $f_k^{(n)} = \mathcal{R}^k(c(\alpha_n, 0))$. We have that $f_0^{(n)}$ tends to $W^s(\mathcal{R}, \Phi)$ when n grow. Then $\{f_k^{(n)}\}_{0\leq k<n}$ attains to $W^s(\mathcal{R},\Phi) \cup W^u(\mathcal{R},\Phi)$ and consequently there exist n_0 s. t. $\{f_k^{(n)}\}_{0\leq k<n} \subset U$, where U is the neighborhood given in conjecture **A**. Therefore the operator $\mathcal{T}_\omega$ is differentiable in the orbit $\{f_k^{(n)}\}_{0\leq k<n} \subset U$. Consider the following hypothesis.

H2) The family $\{c(\alpha,\varepsilon)\}_{(\alpha,\varepsilon)\in A}$ is such that

$$DG_1\left(\omega_{n-1}, f_{n-1}^{(n)}\right) D\mathcal{T}_{\omega_{n-2}}\left(f_{n-2}^{(n)}\right) \cdots D\mathcal{T}_{\omega_0}\left(f_0^{(n)}\right) \partial_\varepsilon c(\alpha_n, 0),$$

has a unique non-degenerate minimum (respectively maximum) as a function from $\mathbb{T}$ to $\mathbb{R}$, for any $n \geq n_0$.

Note that $c(\alpha_n, 0) \in \Sigma_n$, therefore $f_{n-1}^{(n)} \in \Sigma_1$, consequently the function G_1 is defined at the point $f_{n-1}^{(n)}$. Hypothesis **H2** is rather technical and not very intuitive. Further on in this section we show that it is actually satisfied by maps like the Forced Logistic Map.

We have the following result, which ensures the existence of reducibility-loss bifurcations curves in the (α,ε)-plane of parameters near the points $(\alpha_n, 0)$. This is one of the main results of this chapter. On the one hand it gives the existence of reducibility-loss bifurcations, but on the other hand it also gives explicit expression of these bifurcations in term of the renormalization operator $\mathcal{T}_\omega$.

Theorem 3.8. *Consider a family of maps $\{c(\alpha,\varepsilon)\}_{(\alpha,\varepsilon)\in A}$ as before such that hypotheses* **H1** *and* **H2** *are satisfied and consider α_n the parameter values where the uncoupled family intersects the manifolds Σ_n as above. Suppose that the rotation number of the system is equal to $\omega_0 \in \Omega$. Assume also that conjecture* **A** *is true. Then there exists n_0 such that, for any $n \geq n_0$, two curves are born from every parameter value $(\alpha_n, 0)$, locally expressed as $(\alpha_n^+(\varepsilon), \varepsilon)$ and $(\alpha_n^-(\varepsilon), \varepsilon)$, such that they correspond to a reducibility-loss bifurcation of the 2^n-periodic invariant curve.*

Consider also the sequences

$$\begin{array}{rcll} \omega_k & = & 2\omega_{k-1} & \text{for } k = 1,...,n-1. \\ f_k^{(n)} & = & \mathcal{R}\left(f_{k-1}^{(n)}\right) & \text{for } k = 1,...,n-1. \\ u_k^{(n)} & = & D\mathcal{R}\left(f_{k-1}^{(n)}\right) u_{k-1}^{(n)} & \text{for } k = 1,...,n-1. \\ v_k^{(n)} & = & D\mathcal{T}_{\omega_{k-1}}\left(f_{k-1}^{(n)}\right) v_{k-1}^{(n)} & \text{for } k = 1,...,n-1. \end{array} \tag{41}$$

with

$$f_0^{(n)} = c(\alpha_n, 0), \quad u_0^{(n)} = \partial_\alpha c(\alpha_n, 0), \quad v_0^{(n)} = \partial_\varepsilon c(\alpha_n, 0). \tag{42}$$

We also have that

$$\frac{d}{d\varepsilon}\alpha_n^+(0) = -\frac{m\left(DG_1\left(\omega_{n-1}, f_{n-1}^{(n)}\right) v_{n-1}^{(n)}\right)}{D\widehat{G}_1\left(f_{n-1}^{(n)}\right) u_{n-1}^{(n)}}, \tag{43}$$

and

$$\frac{d}{d\varepsilon}\alpha_n^-(0) = -\frac{M\left(DG_1\left(\omega_{n-1}, f_{n-1}^{(n)}\right)v_{n-1}^{(n)}\right)}{D\widehat{G}_1\left(f_{n-1}^{(n)}\right)u_{n-1}^{(n)}}, \tag{44}$$

where m, M, G_1 and $\widehat{G}_1$ are given by equations (31), (32), (30) and (40).

Remark 3.9. If the second part of conjecture **A** is omitted, then we can adapt the result to be asymptotically valid. Recall that we have an open neighborhood of the fixed point Φ where the renormalization operator is differentiable. The uncoupled family $\mathcal{R}^n\left(\{c(\alpha,\varepsilon)\}_{(\alpha,\varepsilon)\in A}\right)$ is contracted towards $W^u(\Phi,\mathcal{R})$, then replacing the family of maps $\{c(\alpha,\varepsilon)\}_{(\alpha,\varepsilon)\in A}$ by $\mathcal{R}^n\left(\{c(\alpha,0)\}_{(\alpha,\varepsilon)\in A}\right)$ it would be close enough to Φ to be differentiable. On the other hand the manifolds Σ_n accumulate to Φ when n goes to ∞, then the functions G_1 and $\widehat{G}_1$ associated to the manifold Σ_1 must be replaced by suitable function G_n and $\widehat{G}_n$ associated to the manifolds Σ_n. These would give place to a more restrictive result, but it would be valid for the asymptotic estimates that are done in [22].

Now we can go back to hypothesis **H2**, which is not intuitive, but we can introduce a stronger condition which is much more easy to check. Moreover this condition is automatically satisfied by maps like the Forced Logistic Map.

Consider a family of maps $\{c(\alpha,\varepsilon)\}_{(\alpha,\varepsilon)\in A}$ as before, satisfying hypothesis **H1**. Consider the following hypothesis on the map.

H2') The family $\{c(\alpha,\varepsilon)\}_{(\alpha,\varepsilon)\in A}$ is such that the quasi-periodic perturbation $\partial_\varepsilon c(\alpha,0)$ belongs to the set

$$\mathcal{B}_1 := \left\{f \in \mathcal{B} |\ f(\theta,x) = u(x)\cos(2\pi\theta) + v(x)\sin(2\pi\theta), \text{ for some } u,v \in \mathcal{RH}(\mathbb{W})\right\}, \tag{45}$$

for any value of α (with $(\alpha,0)\in A$). Here $\mathcal{RH}(\mathbb{W})$ denotes the real holomorphic maps on the set $\mathbb{W}$, and $\mathbb{W}$ is the set given by hypothesis **H0** (see section 2 for more details).

Then we have the following result.

Proposition 3.10. *If a family $\{c(\alpha,\varepsilon)\}_{(\alpha,\varepsilon)\in A}$ satisfies* **H1** *and* **H2'** *then it satisfies* **H2**.

On the other hand, we have the following propositions which allow us to compute explicitly the derivative of G_1 and $\hat{G}_1$.

Proposition 3.11. *Assume that we have $f_1 \in \Sigma_1$. Consider V a neighborhood of f_1 (in the topology of $\mathcal{B}$) and $x : V \to \mathcal{RH}(\mathbb{B}_\rho, \mathbb{W})$ the map such that $x(f)$ is a two periodic invariant curve of f (with rotation number ω). Then we have*

$$[D_f x(f_1)h](\theta) = D_x f_1(1)h(\theta - 2\omega, 0) + h(\theta-\omega, 1).$$

Proposition 3.12. *Assume that we have $f_1 \in \Sigma_1$. Consider V a neighborhood of f_1 (in the topology of $\mathcal{B}$). Consider also the map G_1 defined in (30) where $x(g)$ is the two periodic invariant curve of the map. Then we have*

$$[D_g G_1(f_1)h](\theta) = D_x f_1(1)\Big[D_{x^2}^2 f_1(0)\Big(D_x f_1(1)h(\theta-2\omega,0) + h(\theta-\omega,1)\Big) + D_x h(\theta,0)\Big]. \tag{46}$$

Using the last propositions one can be more explicit on the directions of the reducibility-loss bifurcations given by formulas (43) and (44)

Corollary 3.13. *Assume that the same hypotheses of theorem 3.8 are satisfied and consider additionally the sequences (41) for the same initial terms (42) of the theorem. Then we have*

$$\frac{d}{d\varepsilon}\alpha_n^+(0) = -\frac{\min_{\theta\in\mathbb{T}}\left[\begin{array}{r} D^2_{x^2}f^{(n)}_{n-1}(0)\Big(D_x f^{(n)}_{n-1}(1)v^{(n)}_{n-1}(\theta-2\omega_{n-1},0)+v^{(n)}_{n-1}(\theta-\omega_{n-1},1)\Big) \\ +D_x v^{(n)}_{n-1}(\theta,0)\end{array}\right]}{D_{x^2}f^{(n)}_{n-1}(0)\Big(D_x f^{(n)}_{n-1}(1)u^{(n)}_{n-1}(0)+u^{(n)}_{n-1}(1)\Big)+D_x u^{(n)}_{n-1}(0)}. \quad (47)$$

and

$$\frac{d}{d\varepsilon}\alpha_n^-(0) = -\frac{\max_{\theta\in\mathbb{T}}\left[\begin{array}{r} D^2_{x^2}f^{(n)}_{n-1}(0)\Big(D_x f^{(n)}_{n-1}(1)v^{(n)}_{n-1}(\theta-2\omega_{n-1},0)+v^{(n)}_{n-1}(\theta-\omega_{n-1},1)\Big) \\ +D_x v^{(n)}_{n-1}(\theta,0)\end{array}\right]}{D_{x^2}f^{(n)}_{n-1}(0)\Big(D_x f^{(n)}_{n-1}(1)u^{(n)}_{n-1}(0)+u^{(n)}_{n-1}(1)\Big)+D_x u^{(n)}_{n-1}(0)}, \quad (48)$$

These explicit formulas will be used in [23] to compute numerically the directions $\frac{d}{d\varepsilon}\alpha_n^+(0)$ with the use of a discretization of the operators $\mathcal{R}$ and $\mathcal{T}_\omega$.

Proofs

Proof of theorem 3.8. We consider only the case involving $\frac{d}{d\varepsilon}\alpha_n^+(0)$ since the other is completely analogous. When the hypothesis **H1** is satisfied there exists a sequence of parameter values $\{\alpha_n\}_{n\in Z_+}$, such that the critical point of the map $c(\alpha_n,0)$ is a 2^n-periodic orbit. In other words, we have that $c(\alpha_n,0)\in\Sigma_n$. Moreover the map $c(\alpha_n,0)$ is (at least) $n-1$ times renormalizable in the one dimensional sense. Now, due to the perturbative construction of the q.p. renormalization operator $\mathcal{T}_\omega$ we have that there exists a neighborhood U_n (on $\mathcal{B}$) of $c(\alpha_n,0)$ such that any map in U_n is $n-1$ times renormalizable (in the q.p. sense).

Using theorem 3.5 we have that $\Upsilon_n^+(\omega)$ is locally given as

$$\Upsilon_n^+(\omega)\cap U_n = \{g\in U_n|\, G_1^+(T^{n-1}(\omega,g))=0\},$$

where $G_1^+ = m\circ G_1$, with m the minimum function (31) and G_1 is given by (30).

Consider now a neighborhood $A_n=(a_n,b_n)\times[0,c_n)$ of the parameter value $(\alpha_n,0)$, which is small enough to have $\{c(\alpha,\varepsilon)\}_{(\alpha,\varepsilon)\in A_n}\subset U_n$. We can define the following map

$$\begin{array}{rccc} g_n: & (a_n,b_n)\times[0,c_n) & \to & \mathbb{R} \\ & (\alpha,\varepsilon) & \mapsto & m\circ G_1(T^{n-1}(\omega,c(\alpha,\varepsilon))). \end{array}$$

Then we have that

$$\{c(\alpha,\varepsilon)\}_{(\alpha,\varepsilon)\in A_n}\cap\Upsilon_n^+ = \{(\alpha,\varepsilon)\in A_n|\, g_n(\alpha,\varepsilon)=0\}.$$

The proof of the theorem follows applying the implicit function theorem to the function g_n at the point $(\alpha_n,0)$. With this aim, let us describe $T^{n-1}(\omega,c(\alpha,\varepsilon))$ with some more detail.

Recall that the family $\{c(\alpha,\varepsilon)\}_{(\alpha,\varepsilon)\in A}$ uncouples, therefore for any $(\alpha,\varepsilon)\in A$ we can write $c(\alpha,\varepsilon)=c_0(\alpha)+\varepsilon c_1(\alpha,\varepsilon)$.

From now on assume that (α, ε) are such that $f_k = \mathcal{R}^k(c(\alpha, \varepsilon))$ is contained in the set U of conjecture **A**. Using the differentiability of T (with respect its component on $\mathcal{B}$) we have that

$$T^{n-1}(\omega_0, c(\alpha, \varepsilon)) = (\omega_{n-1}, \mathcal{R}^{n-1}(c_0(\alpha)) + \varepsilon H_0(\omega_0, \alpha, \varepsilon)),$$

with

$$H_0(\omega_0, \alpha, 0) = D\mathcal{T}_{\omega_{n-2}}(f_{n-2}) \cdots D\mathcal{T}_{\omega_0}(f_0)\partial_\varepsilon c(\alpha_n, 0),$$

where $\omega_k = 2^k \omega_0$ and $f_k = \mathcal{R}^k(c(\alpha, 0))$, for $k = 0, \ldots, n-1$.

Applying now the differentiability of G_1 we have

$$G_1(T^{n-1}(\omega_0, c(\alpha, \varepsilon))) = G_1(\omega_{n-1}, \mathcal{R}^{n-1}(c_0(\alpha))) + \varepsilon H_1(\omega_0, \alpha, \varepsilon),$$

with

$$H_1(\omega_0, \alpha, 0) = DG_1(\omega_{n-1}, f_{n-1})H_0(\omega, \alpha, 0),$$

where $\omega_{n-1} = 2^{n-1}\omega_0$ and $f_{n-1} = \mathcal{R}^{n-1}(c(\alpha, 0))$.

Note that $\mathcal{R}^n(c_0(\alpha))$ is an uncoupled map, therefore $G_1\left(\omega_{n-1}, \mathcal{R}^{n-1}(c_0(\alpha))\right)$ as a function of $C^\omega(\mathbb{T}, I)$ is constant, and then its minimum is equal to this constant. Actually we have that $m\left(G_1(\omega_{n-1}, \mathcal{R}^{n-1}(c_0(\alpha)))\right) = \widehat{G}_1(\mathcal{R}^{n-1}(c_0(\alpha))$. On the other hand, note that $\varepsilon \geq 0$ for any $(\alpha, \varepsilon) \in A_n$. Using these two facts to m, the minimum operator (31), we have

$$\begin{aligned}
g_n(\omega, \alpha, \varepsilon) &= m\left(G_1(\omega_{n-1}, \mathcal{R}^{n-1}(c_0(\alpha))) + \varepsilon H_1(\omega_0, \alpha, \varepsilon)\right) \\
&= \widehat{G}_1(\mathcal{R}^{n-1}(c_0(\alpha))) + m\left(\varepsilon H_1(\omega_0, \alpha, \varepsilon)\right) \qquad (49)\\
&= \widehat{G}_1(\mathcal{R}^{n-1}(c_0(\alpha))) + \varepsilon\left(m \circ H_1(\omega_0, \alpha, \varepsilon)\right). \qquad (50)
\end{aligned}$$

Where $\widehat{G}_1$ is defined like in (40) in a suitable neighborhood of $\mathcal{R}^n(c(\alpha))$.

Our aim is to apply the IFT to g_n at the point $(\alpha, \varepsilon) = (\alpha_n, \varepsilon)$. Note that $c(\alpha_n, 0)$ accumulates to $W^s(\Phi, \mathcal{R})$ when n grows. Then the sequence $\{f_k^{(n)}\}_{0 \leq k < n}$ attains to $W^s(\Phi, \mathcal{R}) \cup W^u(\Phi, \mathcal{R})$ when n grows. Therefore there exists n_0 such that, for any $n \geq n_0$, $f_k^{(n)} \in U$ for $k = 0, \ldots, n-1$, where U is the set given by conjecture **A**. Then the function $g_n(\omega, \alpha, \varepsilon)$ is differentiable in a neighborhood of $(\alpha_n, 0)$ if $H_1(\omega_0, \alpha_n, 0))$ has a unique non-degenerate minimum. Note that

$$H_1(\omega_0, \alpha_n, 0) = DG_1(\omega_{n-1}, f_{n-1})D\mathcal{T}_{\omega_{n-2}}(f_{n-2}) \cdots D\mathcal{T}_{\omega_0}(f_0)\partial_\varepsilon c(\alpha_n, 0), \qquad (51)$$

which actually corresponds to the hypothesis **H2**, which is satisfied.

Recall that if $c(\alpha_n, 0) \in \Sigma_n$ then we have $g_n(\omega, \alpha_n, 0) = 0$. Therefore, to apply the Implicit Function Theorem (IFT for short) to g_n, we need to check that $\partial_\alpha g_n(\omega, \alpha_n, 0) \neq 0$. Note also that $\widehat{G}_1$ is the function which gives locally the manifold Σ_1, therefore the condition $\partial_\alpha g_n(\omega, \alpha_n, 0) \neq 0$ is equivalent to require that $\mathcal{R}^{n-1}(c_0(\alpha))_{\alpha \in (a_n, b_n)}$ intersects transversally the manifold Σ_1 at the point $\mathcal{R}^{n-1}(c_0(\alpha_n))$. This condition is satisfied due the hypothesis **H1** which requires the family $\{c(\alpha)\}_{\alpha \in [a,b]}$ to cross transversely each manifold Σ_n for any n.

Then the IFT ensures us that there exists a neighborhood $\tilde{A} = (\tilde{a}_n, \tilde{b}_n) \times [0, \tilde{c_n})$ and a function $\alpha_n(\omega) : [0, d_n) \to R)$ such that

$$\{c(\alpha, \varepsilon)\}_{(\alpha, \varepsilon) \in \tilde{A}} \cap \Upsilon_n(\omega) = \{c(\alpha_n(\omega, \varepsilon), \varepsilon)\}_{\varepsilon \in [0, d_n)}.$$

Moreover we have that

$$\frac{d}{d\varepsilon}\alpha_n^+(0) = -\frac{\partial_\varepsilon g_n(\omega, \alpha_n, 0)}{\partial_\alpha g_n(\omega, \alpha_n, 0)}.$$

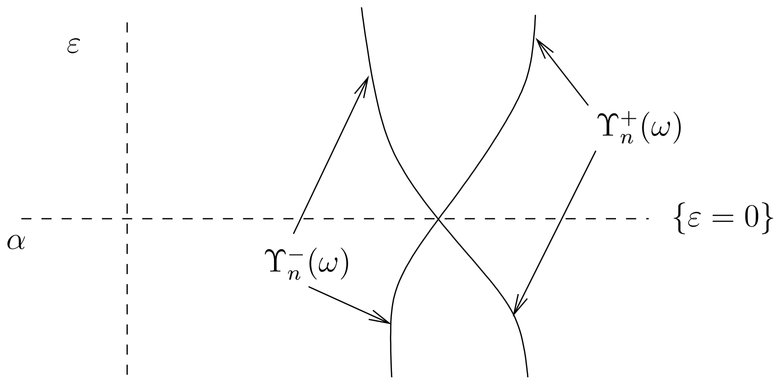

Figure 4: Representation of the intersection of the sets $\Upsilon_n^+(\omega)$ and $\Upsilon_n^-(\omega)$ with a two parametric family of maps. The plane represented correspond to the plane of parameters (α, ε) of the family.

Finally we can use equation (50) and (51) to compute $\partial_\varepsilon g_n(\omega, \alpha_n, 0)$ and $\partial_\alpha g_n(\omega, \alpha_n, 0)$. Then, it follows

$$\frac{d}{d\varepsilon}\alpha_n^+(0) = -\frac{m\left(DG_1\left(\omega_{n-1}, f_{n-1}^{(n)}\right) D\mathcal{T}_{\omega_{n-2}}\left(f_{n-2}^{(n)}\right)\cdots D\mathcal{T}_{\omega_0}\left(f_0^{(n)}\right)\partial_\varepsilon c(\alpha_n, 0)\right)}{D\widehat{G}_1\left(f_{n-1}^{(n)}\right) D\mathcal{R}\left(f_{n-2}^{(n)}\right)\cdots D\mathcal{R}\left(f_0^{(n)}\right)\partial_\alpha c(\alpha_n, 0)},$$

which indeed is equivalent to (43). □

Remark 3.14. In theorem 3.8 the parameter space has been set up in such a way that ε is greater or equal zero for any values of the parameter. In the proof above the IFT is applied to a neighborhood of $(\alpha, \varepsilon) = (\alpha_n, 0)$. But the theorem is not completely applicable in its usual form (see for example [5], because we do not have the derivative defined w.r.t. directions with ε negative. To bypass this difficulty, we can consider the map $g_n(\omega, \alpha, \varepsilon)$ written as in (50). Then we can extend the map symmetrically as

$$g_n(\omega, \alpha, \varepsilon) = \widehat{G}_1(\mathcal{R}^{n-1}(c_0(\alpha))) + \varepsilon\,(m \circ H_1(\omega_0, \alpha, -\varepsilon))\,.$$

for any $\varepsilon < 0$. This extension of the map is enough to have a C^1 map and to apply the IFT. If more differentiability is required, then other extensions should be considered.

Remark 3.15. Recall that the domain of the parameters has been considered of the form $A = [a, b] \times [0, c]$. One might replace this domain for one of the form $\tilde{A} = [a, b] \times [-c, c]$. Then one can redo the proof of theorem 3.8 with the new set up. But then, in the step from (49) to (50) one can not apply the IFT, because the function $g_n(\omega, \alpha, \varepsilon)$ is not differentiable. This is because the minimum is replaced by a maximum for $\varepsilon < 0$ which makes the function only C^0. What one can do is to split $\tilde{A} = [a, b] \times [-c, c]$ into $\tilde{A}_- = [a, b] \times [-c, 0]$ and $\tilde{A}_+ = [a, b] \times [0, c]$. Then one can apply theorem 3.8 twice, and then we obtain four different bifurcation curves emerging form the same point $(\alpha_n, 0)$. What actually happens is that we have two smooth curves, crossing at $(\alpha_n, 0)$, but the curves defined by $\Upsilon_n^+(\omega) \cap \{c(\alpha, \varepsilon)\}_{(\alpha,\varepsilon)\in\tilde{A}}$ and $\Upsilon_n^-(\omega) \cap \{c(\alpha, \varepsilon)\}_{(\alpha,\varepsilon)\in\tilde{A}}$ swap their position when one crosses $\varepsilon = 0$. This is illustrated in figure 4.

Proof of proposition 3.10. Applying proposition 2.16 on section 2, we have that the space $\mathcal{B}_1$ is invariant by $D\mathcal{T}_\omega(f)$ for any $\omega \in \mathbb{T}$ and $f \in \mathcal{B}_0$ in a neighborhood of the fixed point. Consequently, we have that $v_{n-1} = D\mathcal{T}_{\omega_{n-2}}\left(f_{n-2}^{(n)}\right)\cdots D\mathcal{T}_{\omega_0}\left(f_0^{(n)}\right)\partial_\varepsilon c(\alpha_n, 0)$ belongs to $\mathcal{B}_1$, where $\omega_k = 2^k\omega_0$ and $f_k = \mathcal{R}^k(c(\alpha, 0))$. Finally, note that when we consider $DG_1(\omega_{n-1}, f_{n-1})v_{n-1}$

we obtain a periodic function of the form $A\cos(2\pi\theta)+B\sin(2\pi\theta)$, which will have a unique non-degenerate minimum (and maximum) if A and B are not simultaneously zero. $\square$

Proof of proposition 3.11. We have that the map $x : V \to \mathcal{RH}(\mathbb{B}_\rho, \mathbb{W})$ is obtained applying the IFT to the map

$$\begin{array}{rccc} F_1: & V \times \mathcal{RH}(\mathbb{B}_\rho, \mathbb{C}) & \to & \mathcal{RH}(\mathbb{B}_\rho, \mathbb{C}) \\ & (g,x) & \mapsto & [F(g,x)](\theta) := g(\theta+\omega, g(\theta, x(\theta))) - x(\theta+2\omega). \end{array}$$

at the point $(f_1, x(f_1))$. From the IFT we also know that

$$D_h x[f]h = -(D_x F_1(f, x[f]))^{-1} \circ (D_f F_1(f, x[f]))h. \tag{52}$$

Differentiating F_1 we have

$$\begin{array}{rccc} D_x F_1(g, x[g]): & \mathcal{RH}(\mathbb{B}_\rho, \mathbb{C}) & \to & \mathcal{RH}(\mathbb{B}_\rho, \mathbb{C}) \\ & l & \mapsto & D_x g\big(\theta+\omega, g(\theta, x[g](\theta))\big) D_x g\big(\theta, x[g](\theta)\big) l(\theta) - l(\theta+2\omega). \end{array}$$

and

$$\begin{array}{rccc} D_f F_1(g, x[g]): & \mathcal{RH}(\mathbb{B}_\rho \times \mathbb{W}, \mathbb{C}) & \to & \mathcal{RH}(\mathbb{B}_\rho, \mathbb{C}) \\ & h & \mapsto & \begin{array}{c} D_x g\big(\theta+\omega, g(\theta, x[g](\theta))\big) h(\theta, x[g](\theta)) + \\ h\big(\theta+\omega, g(\theta, x[g](\theta))\big). \end{array} \end{array}$$

From the fact that $f_1 \in \mathcal{B}_0$ it follows that its critical point is $x=0$ and $f_1(0)=1$. From $f \in \Sigma_1$ we have that the critical point is a two periodic orbit, therefore $x(f_1)=0$ and $f(f(0))=f(1)=0$. Using these properties, given $h \in \mathcal{RH}(\mathbb{B}_\rho \times \mathbb{W}, \mathbb{C})$ we have

$$\Big[D_f F(f_1, x[f_1])h\Big](\theta) = D_x f_1(1) h(\theta, 0) + h(\theta+\omega, 1).$$

On the other hand, given $\ell \in \mathcal{RH}(\mathbb{B}_\rho \times \mathbb{W}, \mathbb{C})$ we have that

$$\Big[D_x F(f_1, x[f_1])\ell\Big](\theta) = -\ell(\theta+2\omega).$$

Note that this operator is easily invertible, and its inverse is given by

$$\Big[(D_x F_1(f, x[f]))^{-1}\ell\Big](\theta) = -\ell(\theta-2\omega).$$

Now we can use (52) and the last two equations to conclude that

$$\Big[D_h x(f_1)h\Big](\theta) = D_x f_1(1) h(\theta-2\omega, 0) + h(\theta-\omega, 1).$$

$\square$

Proof of proposition 3.12. Using the chain rule on map (30) is it not hard to see that

$$\begin{aligned} \Big[D_g G_1(g)h\Big](\theta) &= D_x g(\theta+\omega, g(\theta, x(\theta)))\Big(D^2_{x^2} g(\theta, x[g](\theta))\big[D_h x(g)h\big](\theta) + D_x h(\theta, x[g](\theta))\Big) \\ &\quad + D_x g(\theta, x[g](\theta)) H(g, x[g], h)(\theta), \end{aligned}$$

where $H(g, x[g], h)$ is an expression on g, $x[g]$ and h that can be explicitly computed but here it has no interest since it will be cancelled out further on.

From the fact that $f_1 \in \mathcal{B}_0$, it follows that its critical point is at $x = 0$ (i.e. $D_x f_1(0) = 0$) and $f_1(0) = 1$. From $f_1 \in \Sigma_1$ we have that the critical point is a two periodic orbit, therefore $x(f_1) = 0$ and $f_1(f_1(0)) = f_1(1) = 0$. Using this properties and the equation above we have

$$\left[D_g G_1(f_1)h\right](\theta) = D_x f_1(1)\Big(D^2_{x^2} f_1(0)\left[D_f x(f_1)h\right](\theta) + D_x h(\theta, 0)\Big).$$

Finally, we can use proposition 3.11 to compute $\left[D_f x(f_1)h\right](\theta)$, then the stated result holds. □

Proof of corollary 3.13. From theorem 3.8 it follows that

$$\frac{d}{d\varepsilon}\alpha_n^+(0) = -\frac{m\left(DG_1\left(\omega_{n-1}, f_{n-1}^{(n)}\right)v_{n-1}^{(n)}\right)}{D\widehat{G}_1\left(f_{n-1}^{(n)}\right)u_{n-1}^{(n)}}. \tag{53}$$

We can apply now proposition 3.12 to compute the derivative of G_1. Note that $\widehat{G}_1$ can be seen as G_1 restricted to $\mathcal{B}_0$, then the proposition 3.12 is also applicable to $\widehat{G}_1$. Then we have

$$\begin{aligned} DG_1\left(\omega_{n-1}, f_{n-1}^{(n)}\right)v_{n-1}^{(n)} = & D_x f_{n-1}^{(n)}(1)\Big[D_{x^2} f_{n-1}^{(n)}(0)\Big(D_x f_{n-1}^{(n)}(1) v_{n-1}^{(n)}(\theta - 2\omega_{n-1}, 0) \\ & + v_{n-1}^{(n)}(\theta - \omega_{n-1}, 1)\Big) + D_x v_{n-1}^{(n)}(\theta, 0), \Big] \end{aligned}$$

and

$$D\widehat{G}_1\left(f_{n-1}^{(n)}\right)u_{n-1}^{(n)} = D_x f_{n-1}^{(n)}(1)\Big[D_{x^2} f_{n-1}^{(n)}(0)\Big(D_x f_{n-1}^{(n)}(1) u_{n-1}^{(n)}(0) + u_{n-1}^{(n)}(1)\Big) + D_x u_{n-1}^{(n)}(0)\Big].$$

Finally let us remark that $f_{n-1}^{(n)}$ belongs to $\mathcal{B}_0$, therefore we have that $D_x f_{n-1}^{(n)}(x)x < 0$ for any $x \in I \setminus \{0\}$. Concretely we have $D_x f_{n-1}^{(n)}(1) < 0$. If we replace the values of $DG_1\left(\omega_{n-1}, f_{n-1}^{(n)}\right)v_{n-1}^{(n)}$ and $D\widehat{G}_1\left(f_{n-1}^{(n)}\right)u_{n-1}^{(n)}$ in (53), then when we simplify the value $D_x f_{n-1}^{(n)}(1) < 0$ the minimum becomes a maximum. □

A The minimum function

In this appendix we give some basic properties of the minimum of a function $f : \mathbb{T} \to \mathbb{R}$ as an operator, with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the one dimensional real torus. To work in the same topology that in the rest of the paper we will consider $f \in \mathcal{RH}(\mathbb{B}_\rho, \mathbb{C})$ the space of real analytic functions from $\mathbb{B}_\rho$ to $\mathbb{C}$, and continuous on the closure of $\mathbb{B}_\rho$, with $\mathbb{B}_\rho$ a band of width ρ around the real torus $\mathbb{T}$. In other words we want to study the operator

$$\begin{array}{rccc} m: & \mathcal{RH}(\mathbb{B}_\rho, \mathbb{C}) & \to & \mathbb{R} \\ & g & \mapsto & \min\limits_{\theta \in \mathbb{T}} g(\theta). \end{array}$$

More concretely we focus on the differentiability of the map m. Note that in the space of holomorphic functions, it has no sense to consider the minimum of a function. Nevertheless,

for maps in $\mathcal{RH}(\mathbb{B}_\rho, \mathbb{C})$ we have that the image of real numbers are real numbers, then we can consider the minimum in the real torus.

Concerning to differentiability of the minimum as an operator, one has the following result.

Proposition A.1. *Let $g_0 \in \mathcal{RH}(\mathbb{B}_\rho, \mathbb{C})$ be a function such that its global minimum in $\mathbb{T}$ is attained only at one value $\theta_0 \in \mathbb{T}$ and it is not degenerate, i.e. $g''(\theta_0) > 0$. Then there is a function $G : \mathcal{RH}(\mathbb{B}_\rho, \mathbb{C}) \to \mathbb{C}$ such that $G(g) = m(g)$ for any $g \in \mathcal{RH}(\mathbb{B}_\rho, \mathbb{C})$ and G is infinitely many times differentiable in a small neighborhood of g_0. In other words we have that m can be extended to a differentiable function $G : \mathcal{RH}(\mathbb{B}_\rho, \mathbb{C}) \to \mathbb{C}$ around g_0.*

Moreover, the derivative of G in g_0 is given by

$$DG(g_0)\, g_1 = g_1(\theta_0).$$

Proof. To prove the proposition we construct the map G using the IFT.

Consider the auxiliary function

$$\begin{array}{rccc} F: & \mathcal{RH}(\mathbb{B}_\rho, \mathbb{C}) \times \mathbb{T} & \to & \mathbb{C} \\ & (g, \theta) & \mapsto & g'(\theta). \end{array}$$

We have that the function F corresponds to the derivative of g composed with the evaluation in the point θ. Both the derivative and the evaluation map are C^∞ functions (they are bounded linear operators, therefore they are infinitely differentiable), then its composition, which is F, is also C^∞.

Consider g_0 as in the hypothesis of the proposition. Then we have that there exist a $\theta_0 \in \mathbb{T}$ such that (g_0, θ_0) is a zero of F. By hypothesis we have that θ_0 is not a degenerate minimum, then

$$D_\theta F(g_0, \theta_0) = g_0''(\theta_0) > 0,$$

We can apply now the IFT, consequently we have that there exist an open neighborhood $U \subset \mathcal{RH}(\mathbb{B}_\rho, \mathbb{C})$ of g_0 and a C^∞ function

$$\begin{array}{rccc} \theta: & U & \to & \mathbb{T} \\ & g & \mapsto & \theta(g), \end{array} \tag{54}$$

such that $F(g, \theta(g)) = 0$ for any $g \in U$. Concretely we have that $\theta(g)$ is a local minimum for any $g \in U$. Recall that by hypothesis we have also that θ_0 is a global minimum of g_0 and that it is unique. Then, reducing U to a smaller neighborhood if necessary, we have that $\theta(g)$ is also the unique global minimum of g, for any $g \in U$.

Let us consider the evaluation map $\mathrm{Ev} : \mathcal{RH}(\mathbb{B}_{\rho'}, \mathbb{C}) \times \mathbb{T} \to \mathbb{C}$ the evaluation map (in a point of the real torus $\mathbb{T} \subset \mathbb{B}_{\rho'}$) with ρ' a value $0 < \rho' < \rho$. Let θ be the function (54) defined by the IFT. For any $g \in U$ we can define the map G in the statement of the proposition A.1 as $G(g) = \mathrm{Ev}(g', \theta(g))$. With this definition we have that $G(g) = m(g)$ for any $g \in U$. Therefore the minimum is a C^∞ function in a neighborhood of g_0. Note that in the definition of the evaluation map Ev we have considered $\mathcal{RH}(\mathbb{B}_{\rho'}, \mathbb{C})$ as its domain with $0 < \rho' < \rho$. This is needed in order to ensure that g' is a bounded function, then one has G is a bounded operator.

To finish we compute the derivative of the minimum function. Is not difficult to see that $D_\theta Ev(f, \theta) = f'(\theta)$ and $D_g Ev(f, \theta) g_1 = g_1(\theta)$. Using the chain rule we have

$$DG(g)g_1 = D_g Ev(g, \theta(g))g_1 + D_\theta Ev(g, \theta(g)) D_g\theta(g) g_1.$$

Recall that $g_0'(\theta(g_0)) = g_0'(\theta_0) = 0$, then we have $D_\theta Ev(g, \theta(g)) = g'(\theta(g)) = 0$. Then it follows

$$DG(g_0)\, g_1 = g_1(\theta_0).$$

□

Let us discuss the case when the hypotheses of the proposition A.1 are not satisfied. The two main hypotheses of the proposition are the non-degeneracy of the minimum and the uniqueness of it. When the non-degeneracy condition is suppressed the proposition still being true, since the argument done before can be adapted changing the auxiliary function, although the proof becomes quite more technical.

On the other hand, the uniqueness condition of the minimum is always necessary, because when it is not satisfied the map ceases to be Frechet differentiable.

We just want to see that it is not differentiable. For simplicity we will take de derivative in the reals. Let us consider the most degenerate case, which is when the function g_0 is constant. Then given $g_1 \in \mathcal{RH}(\mathbb{B}_\rho, \mathbb{C})$ we want to compute $Dm(g_0, g_1)$, the Gateaux derivative of m at g_0 with respect to the g_1 direction. By definition we have

$$Dm(g_0, g_1) = \lim_{t \to 0} \frac{m(g_0 + tg_1) - m(g_0)}{t}$$

whenever the limit exist.

Indeed, when the function $g_0(\theta)$ is constant we have that

$$\min_{\theta \in T}(g_0 + tg_1) = g_0 + \min_{\theta \in T} tg_1 = \begin{cases} g_0 + t \min\limits_{\theta \in T} g_1 & \text{if } t \geq 0 \\ \\ g_0 + t \max\limits_{\theta \in T} g_1 & \text{if } t \leq 0 \end{cases}$$

Then the Gateaux derivative not only depends on the sign of t when taking the limit, moreover when the sign is fixed the limit which we obtain is not even a linear operator. It is clear that in this case the minimum operator is not differentiable.

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